JOURNAL OF APPROXIMATION THEORY 73, 17–92 (1993)

Uniqueness in Vector-Valued Approximation

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Communicated by Will Light

Received January 23, 1991; accepted December 31, 1991

We consider when a finite dimensional subspace is a unicity space in the problem of best vector-valued approximation. Different choices of mixed norms are discussed.

1. INTRODUCTION

This paper grew out of an attempt to survey the topic of vector-valued approximation. Rather quickly we realized that this subject had been considered in only a handful of papers, and that numerous questions remained unresolved. Some we were able to answer to our satisfaction, while others remain unanswered. We hope that this paper will further stimulate work in this area.

Let X be a normed linear space and Y a subset of X. Numerous authors mention four basic questions of qualitative approximation theory (see, e.g., Garkavi [9], de Boor [8], Light and Cheney [24]). These are the questions of existence, characterization, uniqueness, and construction of a best approximant to elements of X from Y. In this paper we concern ourselves with the questions of characterization and uniqueness. Our main interest is in the uniqueness question. Existence will always hold since we consider approximation from finite dimensional subspaces. Concerning construction we have nothing to say, and in fact little seems to be known.

By vector-valued functions we mean

$$\mathbf{f}(x) = (f_1(x), ..., f_m(x)), \qquad x \in D,$$

where D is some set and each $f_i\colon D \to \mathbb{R}$. In other words $\mathbf{f}\colon D \to \mathbb{R}^m$. We essentially look at two clases of simple mixed norms on such functions. These are

$$\|\mathbf{f}\|_{A(p,q)} = \left(\sum_{i=1}^{m} \left(\int_D |f_i(x)|^q \, dv(x) \right)^{p/q} \right)^{1/p}$$

0021-9045/93 $5.00

and

$$\|\mathbf{f}\|_{B(p,q)}=\left(\int_D\left(\sum_{i=1}^m |f_i(x)|^q\right)^{p/q} d\nu(x)\right)^{1/p},$$

where $p, q \in [1, \infty]$ (with the usual understanding if $p=\infty$ and/or $q=\infty$). We are more explicit and accurate in subsequent sections. These are special cases of

$$\|\mathbf{f}\|_A = \|\{\|f_i\|_S\}_{i=1}^m\|_T$$

and

$$\|\mathbf{f}\|_B = \| \ \|\mathbf{f}(x)\|_T\|_S,$$

where $\|\cdot\|_T$ is any norm on $\mathbb{R}^m$ and $\|\cdot\|_S$ any norms on the "appropriate" function spaces.

We look at $\|\cdot\|_{A(p,q)}$ and $\|\cdot\|_{B(p,q)}$ for each $p, q \in [1, \infty]$ and attempt to answer the following question. Given a finite dimensional subspace U, what are conditions on U such that to each $\mathbf{f}$ there exists a unique best approximant from U? Towards this end we generally are forced to consider the question of the characterization of best approximants. There are many different cases depending on A and B, and whether $p=1$, $1<p<\infty$, $p=\infty$ and $q=1$, $1<q<\infty$, $q=\infty$. Some of these are easily dealt with. The remaining cases are dealt with in some detail. This is the reason for the length of this paper. We also consider as special cases the problems of Simultaneous Approximation and Tensor Product Approximation. By Simultaneous Approximation we mean approximation from subspaces where each of the approximating functions has the form

$$\mathbf{u}(x) = (u(x), ..., u(x)),$$

i.e., $u_i(x) = u(x)$ for each $i=1, ..., m$. By Tensor Product Approximation we mean approximation from subspaces which contain a basis of functions, all of the whose components are identically zero except for one non-trivial component.

In Section 2 we give a series of general results concerning characterizing best approximations and uniqueness. These results are all known, but it is well worth quickly reviewing them as they are relevent in our subsequent analysis.

In Section 3 we present a quick review of some results on unicity spaces in the C and L^1 norms, as these are the basic non-smooth, non-strictly convex norms considered. Sections 4–12 represent Part A and are concerned with the $A(p, q)$-norm. In Sections 13–19 (Part B) we deal with the $B(p, q)$-norm.

As mentioned previously, we had originally intended to survey for ourselves the topic of vector-valued approximation, but were surprised by the lack of results to the found. Two major exceptions to this are the papers by Zuhovitsky and Stechkin [36] and by Kroó [21] (see also the references therein). The paper by Zuhovitsky and Stechkin essentially covers the cases of the $A(\infty, q)$ and $B(\infty, q)$-norms, $1 < q < \infty$, Sections 7 and 16. This paper is fairly well known in the former Soviet Union, but less so in the west. The paper by Kroó deals with many of the results found in Section 14 on the $B(1, q)$-norm, $1 < q < \infty$. Some related and more specific questions have been dealt with, for example, by Brannigan [3], Garkavi [10], Kroó [18], and Opfer [26]. Some corresponding results on mixed-norm best approximation may be found in Cheney, McCabe, and Phillips [6] and Watson [35].

2. General Results

There are two basic approaches to characterization theorems in the problem of best approximation from linear subspaces. The first of these is based on functional analytic methods. The other is, in spirit at least, a more classical approach and is based on a "generalized perturbation technique." We quickly reviw these two approaches, starting with the former.

Let X be a normed linear space with norm $\|\cdot\|_X$. By X^* we denote the continuous dual of X with associated induced norm $\|\cdot\|_{X^*}$. Let $S(X^*)$ denote the unit ball in X^*. We then have the following characterization of best approximants from linear subspaces.

Theorem 2.1. *Let U be a linear subspace of X and $f \in X \setminus \bar{U}$. Then $u^* \in U$ is a best approximant to f from U if and only if there exists an $h \in X^*$ satisfying*

(1) $\|h\|_{X^*} = 1$

(2) $h(u) = 0$, *all* $u \in U$

(3) $h(f - u^*) = \|f - u^*\|_X$.

The proof of this theorem is simple and may be found, for example, in Singer [33, p. 18] (see also Buck [4]). The more "difficult" part of the proof is a simple application of the Hahn–Banach Theorem and was known to Banach, see, e.g., [2, p. 57].

For specific examples, as we shall see, the difficulties encountered in applying Theorem 2.1 are generally the identification of X^*, and the possible $h \in X^*$ which satisfy (1), (2), and (3). The latter problem is considerably easier if, for example, X is *smooth*. That is, if to each $f \in X$, $f \neq 0$, there exists a unique $h \in X^*$ satisfying $\|h\|_{X^*} = 1$ and $h(f) = \|f\|_X$. In this case the h of Theorem 2.1 is uniquely defined by (1) and (3), and as such, is generally simpler to determine.

A strengthened form of Theorem 2.1 is available if U is of finite dimension. Before stating this strengthened form, we recall that y is an *extreme point* of a convex set B if $\lambda y_1 + (1-\lambda) y_2 = y$ for some $\lambda \in (0, 1)$ and $y_1, y_2 \in B$ implies that $y_1 = y_2 = y$. This next result is due to Singer [33, p. 170] and is a consequence of an elegant application of the Krein–Milman and Alaoglu Theorems.

THEOREM 2.2. *Let X be a normed linear space over the reals, and U an n-dimensional subspace of X. Given $f \in X \setminus U$, we have that u^* is a best approximant to f from U if and only if for some k, $1 \leqslant k \leqslant n+1$, there exist $\lambda_i > 0$, $i = 1, \ldots, k$, and h_i, extreme points of $S(X^*)$, $i = 1, \ldots, k$, such that*

(a) $\sum_{i=1}^{k} \lambda_i h_i(u) = 0$, *all* $u \in U$.

(b) $h_i(f - u^*) = \|f - u^*\|_X$, $i = 1, \ldots, k$.

If X is smooth, then this result adds no new information to Theorem 2.1 since the h_i's are all then equal to the unique h satisfying (1) and (3) of Theorem 2.1.

The second general approach to characterization theorems is based on the idea of directional derivatives for convex functions. The convex function in this case is the norm. Given $f, g \in X$, we define

$$\tau_+(f, g) = \lim_{t \to 0^+} \frac{\|f + tg\|_X - \|f\|_X}{t}. \tag{2.1}$$

The functional τ_+ exists for any $f, g \in X$. This follows from the fact that the quantity

$$\frac{\|f + tg\|_X - \|f\|_X}{t}$$

is both non-decreasing and bounded below on $(0, \infty)$. The two-sided limit in (2.1) need not exist. It exists for *every* $f, g \in X$, $f \neq 0$, if and only if X is smooth. In general $\tau_+(f, g)$, $f \neq 0$, is the supremum of $h(g)$ as h ranges over all norm one linear functionals in X^* satisfying $h(f) = \|f\|_X$, see, e.g., Köthe [15, p. 349]. The two-sided limit, if it exists, is called the Gateaux derivative of f in the direction g. As such, we refer to $\tau_+(f, g)$ as the onesided Gateaux derivative (of f in the direction g). We have, see, e.g., Pinkus [29, p. 3],

THEOREM 2.3. *Let U be a linear subspace of X and $f \in X \setminus \bar{U}$. Then $u^* \in U$ is a best approximant to f from U if and only if $\tau_+(f-u^*, u) \geq 0$ for all $u \in U$.*

As noted, this concept adds little for smooth spaces. Two classic non-smooth spaces for which τ_+ is well known are the following.

(a) Let D be a compact Hausdorff set and $C(D)$ the space of continuous real-valued functions defined on D with norm

$$\|f\|_\infty = \max_{x \in D} |f(x)|.$$

Then for $f, g \in C(D)$, $f \neq 0$,

$$\tau_+(f, g) = \max_{x \in A} [g(x) \operatorname{sgn}(f(x))],$$

where $A = \{x : |f(x)| = \|f\|_\infty\}$.

(b) Let D be a set, Σ a σ-field of subsets of D, and ν a positive measure on Σ. By $L^1(D, \Sigma, \nu)$ we mean the usual space of real-valued ν-measurable functions f defined on D for which $|f|$ is ν-integrable, and

$$\|f\|_1 = \int_D |f| \, d\nu.$$

For $f \in L^1(D, \Sigma, \nu)$, we set

$$Z(f) = \{x : f(x) = 0\}.$$

$Z(f)$ is ν-measurable. For $f, g \in L^1(D, \Sigma, \nu)$, $f \neq 0$, it follows that

$$\tau_+(f, g) = \int_D g(\operatorname{sgn} f) \, d\nu + \int_{Z(f)} |g| \, d\nu,$$

where

$$\operatorname{sgn}(f(x)) = \begin{cases} 1, & f(x) > 0 \\ 0, & f(x) = 0 \\ -1, & f(x) < 0. \end{cases}$$

Given a finite dimensional subspace U of X, is the best approximant to f from U necessarily unique for all $f \in X$? If U enjoys this property, we say that U is a *unicity* space. As is both well known and very easily shown, if the normed linear space X is strictly convex, then U is a unicity space. In general there is no other good criterion for determining when each $f \in X$ has a unique best approximant from U. One theorem found in the literature (see, e.g., Singer [33, p. 104], Holland and Sahney [12, p. 105]) is:

THEOREM 2.4. *Let U be a linear subspace of X. To each element of X there exists at most one best approximant from U if and only if there do not exist $f_1, f_2 \in X$ and $h \in X^*$ satisfying*

(1) $f_1 - f_2 \in U \setminus \{0\}$

(2) $\|h\|_{X^*} = 1$

(3) $h(u) = 0$, *all* $u \in U$

(4) $h(f_1) = \|f_1\|_X$, $h(f_2) = \|f_2\|_X$.

It is our view that this theorem is essentially a tautology and provides no insight into the problem of uniqueness. The proof of Theorem 2.4 comes from a simple application of Theorem 2.1.

3. CHARACTERIZATION AND UNIQUENESS IN C AND L^1

In this section we present a quick review of various facts needed in the subsequent analysis. The material in (A) may be found in many of the standard texts in approximation theory, see, e.g., Cheney [5], Singer [33]. The material of (B) may be found in Pinkus [29].

(A) Let D be a compact Hausdorff set and $C(D)$ the space of continuous real-valued functions defined on D with norm

$$\|f\|_\infty = \max_{x \in D} |f(x)|.$$

From Theorem 2.3 we have:

PROPOSITION 3.1. *Let U be a linear subspace of $C(D)$ and $f \in C(D) \setminus \bar{U}$. Then $u^* \in U$ is a best approximant to f from U if and only if*

$$\max_{x \in A} u(x) \operatorname{sgn}((f - u^*)(x)) \geqslant 0 \tag{3.1}$$

for all $u \in U$, where $A = \{x : |(f - u^*)(x)| = \|f - u^*\|_\infty\}$.

Inequality (3.1) is generally referred to as Kolmogorov's criterion. If U is finite dimensional, then as an application of Carathéodory's Theorem to (3.1), or more directly as a consequence of Theorem 2.2, we obtain the classic characterization theorem on $C(D)$.

THEOREM 3.2. *Let U be an n-dimensional subspace of $C(D)$. Given $f \in C(D)$, we have that u^* is a best approximant to f from U if and only if for some k, $1 \leqslant k \leqslant n+1$, there exist points $\{x_i\}_{i=1}^k \subseteq D$, and real numbers $c_i \neq 0$, $i = 1, ..., k$, such that*

(1) $\sum_{i=1}^k c_i u(x_i) = 0$, *all* $u \in U$

(2) $(\operatorname{sgn} c_i)((f - u^*)(x_i)) = \|f - u^*\|_\infty$, $i = 1, ..., k$.

The finite dimensional unicity spaces in $C(D)$ were characterized by Haar [11].

THEOREM 3.3. *An n-dimensional subspace U of $C(D)$ is a unicity space if and only if no $u \in U \setminus \{0\}$ has more than $n-1$ distinct zeros on D.*

Subspaces satisfying the above condition are called *Haar spaces*. One often sees an equivalent definition of Haar spaces in terms of non-vanishing of certain determinants. Haar spaces on intervals of $\mathbb{R}$ are called *Chebyshev* or *T-systems*. The condition of being a Haar space is rather demanding. For $n > 1$, Haar spaces do not live on domains not homeomorphic to subsets of S^1 (the circle).

(B) Let D be a set, Σ a σ-field of subsets of D, and ν a positive σ-finite measure defined on Σ. Let $L^1(D, \nu) = L^1(D, \Sigma, \nu)$ be as defined in Section 2. From either Theorem 2.1 or 2.3 we have:

THEOREM 3.4. *Let U be a linear subspace of $L^1(D, \nu)$ and $f \in L^1(D, \nu)$. Then u^* is a best approximant to f from U if and only if*

$$\left| \int_D u \operatorname{sgn}(f - u^*)\, d\nu \right| \leqslant \int_{Z(f-u^*)} |u|\, d\nu \tag{3.2}$$

for all $u \in U$.

(At times, as we shall see, it will actually be more convenient to work with the characterization in Theorem 2.1 rather than (3.2).)

With regards to the question of unicity spaces in $L^1(D, \nu)$, there is a fundamental difference depending on whether ν has atoms or does not. We only consider the case where ν is *non-atomic*. The following result for $D = [0, 1]$ and Lebesgue measure is due to Krein [16]. This general form was proved by Phelps [27], see also Moroney [25].

THEOREM 3.5. *Let ν be a non-atomic positive measure. No finite dimensional subspace U of $L^1(D, \nu)$ is a unicity space for $L^1(D, \nu)$.*

In what follows, U is always assumed to be of finite dimension. If we restrict ourselves only to the space of continuous functions, rather than all L^1 functions with this same norm, then it may well be that there are unicity spaces. Towards this end, we let K denote a compact subset of $\mathbb{R}^d$ satisfying $K=\overline{\operatorname{int} K}$, and μ any non-atomic, positive finite measure on K with the property that every real-valued $f \in C(K)$ is μ-measurable, and such that if

$$\|f\|_1 := \int_K |f(x)| \, d\mu(x) = 0$$

for $f \in C(K)$, then $f=0$, i.e., $\|\cdot\|_1$ is truly a norm on $C(K)$. For notational ease we denote the set of such measures by $\mathscr{A}$, and we let $C_1(K, \mu)$ denote the linear space $C(K)$ equipped with norm $\|\cdot\|_1$. $C_1(K, \mu)$ is a normed linear space, but it is not complete.

That unicity spaces for $C_1(K, \mu)$ exist is well known from Jackson's Theorem [14] from 1921, which says that for $K=[0,1]$ and $d\mu = dx$, Lebesgue measure, the algebraic polynomials of any fixed degree are unicity spaces in $C_1([0,1], dx)$. Two characterizations of unicity spaces in $C_1(K, \mu)$ are known. The first is due to Cheney and Wulbert [7], the second to Strauss [34].

THEOREM 3.6. *U is a unicity space for $C_1(K, \mu)$ if and only if there does not exist an $h \in L^\infty(K, \mu)$ and a $u^* \in U$, $u^* \neq 0$, for which*

(1) $|h(x)| = 1$, *all* $x \in K$

(2) $\int_K hu \, d\mu = 0$, *all* $u \in U$

(3) $h\,|u^*| \in C(K)$.

THEOREM 3.7. *U is a unicity space for $C_1(K, \mu)$ if and only if the zero function is not a best $L^1(K, \mu)$-approximant from U to any $g \in U^*$, $g \neq 0$, where*

$$U^* = \{ g : g \in C(K),\ |g| = |u| \text{ for some } u \in U \}.$$

It has been noted that the various necessary and sufficient conditions delineated for U to be a unicity space are μ dependent. That is, U may be a unicity space for $C_1(K, \mu)$ for some measure μ, and not a unicity space for other measures μ. As such, it is natural to ask for necessary and sufficient conditions on U implying that it is a unicity set for $C_1(K, \mu)$ for *all* "nice" measures μ. This problem has been considered in Kroó [19] and Pinkus [28].

We explain the results obtained. For each $u \in U$, $u \neq 0$, the (relatively) open set $K \backslash Z(u)$ is the union of a possibly infinite, but necessarily countable number of open disjoint connected subsets of K, i.e., $K \backslash Z(u) = \bigcup_{i=1}^r A_i$, where the A_i are open, disjoint, and connected. For convenience we also introduce the following notation. $[K \backslash Z(u)]$ denotes the number of open connected disjoint components of $K \backslash Z(u)$, and for each $u \in U$,

$$U(u) = \{v : v \in U, v = 0 \text{ a.e. on } Z(u)\},$$

where the a.e. (almost everywhere) is with respect to Lebesgue measure.

We say that U satisfies *Property* A if to each $u \in U$, $u \neq 0$, with $K \backslash Z(u) = \bigcup_{i=1}^{r} A_i$, as above, and to each choice of $\varepsilon_i \in \{-1, 1\}$, $i = 1, ..., r$, there exists a $v \in U(u)$, $v \neq 0$, satisfying $\varepsilon_i v \geqslant 0$ on A_i, $i = 1, ..., r$. We can also state Property A in two other equivalent forms. Namely, U satisfies Property A if to each $g \in U^*$, $g \neq 0$, there exists a $u \in U$, $u \neq 0$, satisfying $u = 0$ a.e. on $Z(g)$ and $ug \geqslant 0$. Alternatively, U satisfies Property A if to each $u \in U$, $u \neq 0$, and $h \in L^\infty(K)$ with

(1) $|h(x)| = 1$ all $x \in K$

(2) $h\,|u| \in C(K)$

there exists a $v \in U$, $v \neq 0$, satisfying $hv \geqslant 0$.

The following result was proven with restrictions in Kroó [19] and Pinkus [28], and in this form in Kroó [20]. Schmidt [32] later proved a somewhat more general result.

THEOREM 3.8. *U is a unicity space for $C_1(K, \mu)$ for all $\mu \in \mathscr{A}$ if and only if U satisfies Property* A.

This result naturally raises the question of which subspaces satisfy Property A. We know of two necessary conditions implied by Property A. To explain one of these conditions, we say that *U decomposes* if there exist non-trivial subspaces V and W of U such that $U = V \oplus W$, i.e., $U = V + W$ and $V \cap W = 0$, such that $(K \backslash Z(v)) \cap (K \backslash Z(w)) = \varnothing$ for all $v \in V$ and $w \in W$. In other words, there exist disjoint subsets B and C of K such that every function in V vanishes identically off B, while every function of W vanishes identically off C. Two necessary conditions for U to satisfy Property A were given in Pinkus and Wajnryb [30].

THEOREM 3.9. *If U satisfies Property* A, *then*

(1) $[K \backslash Z(u)] \leqslant \dim U(u)$, *for all* $u \in U$.

(2) *If* $Z(U) = \bigcap_{u \in U} Z(u)$, *and* $[K \backslash Z(U)] \geqslant 2$, *then U decomposes.*

If $K \subset \mathbb{R}$, then based on these results a full characterization of those U satisfying Property A may be given. From (2) of Theorem 3.9, it suffices to state this result for $K = [a, b]$ under the assumption that $Z(U) \cap (a, b) = \varnothing$. This next result was proved by Pinkus [28, 29] and improved by Li [22].

THEOREM 3.10. *Let U be a finite-dimensional subspace of $C[a, b]$. Assume $Z(U) \cap (a, b) = \varnothing$. Then the following are equivalent.*

(1) *U satisfies Property A.*

(2) *$[[a, b] \backslash Z(u)] \leqslant \dim U(u)$ for all $u \in U$.*

(3) *U is a WT-system, and if $u \in U$ vanishes on $[c, d]$, $a < c < d < b$, then there exists a $v \in U$ such that $v = u$ on $[a, c]$, and $v = 0$ on $[c, b]$.*

(The subspace U is a WT (weak Chebyshev) system on $[a, b]$ if no $u \in U$ has more than $\dim U - 1$ sign changes on $[a, b]$.) For $K \subset \mathbb{R}^d$, $d \geqslant 2$, such a characterization is not yet known.

PART A

4. THE $A(p, q)$-NORM: GENERAL RESULTS

The next 8 sections contain uniqueness and characterization results with regards to the $A(p, q)$-norm

$$\|\mathbf{f}\|_{A(p, q)} = \left(\sum_{i=1}^{m} \left(\int_D |f_i(x)|^q \, dv(x) \right)^{p/q} \right)^{1/p}, \tag{4.1}$$

where $p, q \in [1, \infty]$ (with the usual understanding if $p = \infty$ and/or $q = \infty$), and generalizations thereof. The $A(p, q)$-norms are, in a sense, conceptually simpler than the $B(p, q)$-norms. They can and sometimes should be considered as spaces of real-valued, rather than vector-valued functions. This is done by setting $f = f_i$ on D_i, for $i = 1, ..., m$ (where here each $D_i = D$, but we think of them as different). That is, f is a function defined on $\bigcup_{i=1}^m D_i$, where $f|_{D_i} = f_i$. The $A(p, q)$-norm of $\mathbf{f}$ is then a particular mixed (p, q)-norm defined on f. (The $B(p, q)$-norm might be thought of in this way, but it is less useful.) Before dealing with specific (p, q), some general remarks are in order.

Let $f: S \times T \to \mathbb{R}$. Assume that for each fixed $t \in T$, $f(\cdot, t)$ is an element of the normed linear space on S with norm

$$\|f(\cdot, t)\|_S,$$

and $\|f(\cdot, t)\|_S$, as a function of t, is an element of a normed linear space on T. It is *not* necessarily true that the quantity

$$\| \, \|f(\cdot, t)\|_S \|_T \tag{4.2}$$

is a norm. It is a norm if T is a lattice and $\|\cdot\|_T$ is a monotone norm. That is, if for any x, y in the normed linear space on T satisfying $0 \leqslant x(t) \leqslant y(t)$

for all t, we necessarily have $\|x\|_T \leqslant \|y\|_T$, then (4.2) is a norm. Thus the $A(p,q)$-norm of (4.1) (and the $B(p,q)$-norm as given in Section 1) are indeed norms. Because of the simple nature of the $A(p,q)$-norm, its dual and the extreme points thereof are easily identified. In fact, let us assume that we are given

$$\mathbf{f}(x)=(f_1(x),\ldots,f_m(x)), \qquad x\in D,$$

normed linear spaces $\tilde{S}_i$, with norms $\|\cdot\|_{S_i}$ on f_i, and a Banach lattice $\tilde{T}$ with norm $\|\cdot\|_T$ over $\mathbb{R}^m$. Let $\tilde{A}$ be the normed linear space with associated norm $\|\cdot\|_A$ given by

$$\|\mathbf{f}\|_A=\|\{\|f_i\|_{S_i}\}_{i=1}^m\|_T. \tag{4.3}$$

Then it is easily seen that the dual space $\tilde{A}^*$ is given by

$$\tilde{A}^*=\{\mathbf{h}:\mathbf{h}=(h_1,\ldots,h_m),\, h_i\in\tilde{S}_i^*\}$$

with norm

$$\|\mathbf{h}\|_{A^*}=\|\{\|h_i\|_{S_i^*}\}_{i=1}^m\|_{T^*}$$

(where $\|\cdot\|_{S_i^*}$ is the norm on $\tilde{S}_i^*$, and $\|\cdot\|_{T^*}$ the norm on $\tilde{T}^*$), and

$$\mathbf{h}(\mathbf{f})=\sum_{i=1}^m h_i(f_i).$$

Thus for $A(p,q)$ as above, we have

$$A^*(p,q)=A(p',q')$$

if $1\leqslant p\leqslant\infty$, $1\leqslant q<\infty$ (where $1/p+1/p'=1/q+1/q'=1$). The case $q=\infty$ is excluded because of the nature of $(L^\infty)^*$. In the more general case of $\tilde{A}$ and $\tilde{A}^*$, we have:

PROPOSITION 4.1. *The element* $\mathbf{h}=(h_1,\ldots,h_m)$ *is an extreme point of* $S(\tilde{A}^*)$ *if and only if*

(a) *each* h_i *is an extreme point of the ball* $\|h_i\|_{S_i^*}S(\tilde{S}_i^*)$,

(b) $\{\|h_i\|_{S_i^*}\}_{i=1}^m$ *is an extreme point of* $S(\tilde{T}^*)$.

Proof. ($\Rightarrow$) Assume $\mathbf{h}$ is an extreme point of $S(\tilde{A}^*)$.

(a) If h_i is not an extreme point of the ball $\|h_i\|_{S_i^*}S(\tilde{S}_i^*)$ for some $i\in\{1,\ldots,m\}$, then there exist $h_i^1, h_i^2\in\tilde{S}_i^*$, $h_i^1\neq h_i^2$ satisfying

$$\|h_i^k\|_{S_i^*}=\|h_i\|_{S_i^*}, \qquad k=1,2$$

and

$$h_i = (h_i^1 + h_i^2)/2.$$

Set

$$\mathbf{h}^k = (h_1^k, \ldots, h_m^k), \qquad k = 1, 2,$$

where $h_j^1 = h_j^2 = h_j$, $j \neq i$, and h_i^1, h_i^1 are as above. It follows that $\mathbf{h}^1 \neq \mathbf{h}^2$,

$$\mathbf{h} = (\mathbf{h}^1 + \mathbf{h}^2)/2$$

and $\|\mathbf{h}^1\|_{A^*} = \|\mathbf{h}^2\|_{A^*} = 1$. Thus $\mathbf{h}$ is not an extreme point of $S(\tilde{A}^*)$.

(b) Assume $\{\|h_i\|_{S_i^*}\}_{i=1}^m$ is not an extreme point of $S(\tilde{T}^*)$. Set $c_i = \|h_i\|_{S_i^*}$, $i = 1, \ldots, m$. Then by assumption, there exist $\mathbf{c}^1, \mathbf{c}^2 \in S(\tilde{T}^*)$, $\mathbf{c}^1 \neq \mathbf{c}^2$, such that

$$\mathbf{c} = (\mathbf{c}^1 + \mathbf{c}^2)/2.$$

Set

$$h_i^k = \frac{c_i^k h_i}{c_i}, \qquad i = 1, \ldots, m; k = 1, 2$$

if $c_i > 0$. If $c_i = 0$, then $c_i^1 = -c_i^2$ and we let $h_i^1 \in \tilde{S}_i^*$ satisfy $\|h_i^1\|_{S_i^*} = |c_i^1|$ and $h_i^2 = -h_i^1$. Thus

$$\|h_i^k\|_{S_i^*} = |c_i^k|, \qquad i = 1, \ldots, m; k = 1, 2.$$

Since $\tilde{T}$ is a Banach lattice on $\mathbb{R}^m$ with respect to the usual elementwise order, it follows that $\tilde{T}^*$ has this same property. Thus

$$1 \geqslant \|\mathbf{c}^k\|_{T^*} = \| \ |\mathbf{c}^k| \ \|_{T^*} = \|\{\|h_i^k\|_{S_i^*}\}_{i=1}^m\|_{T^*}.$$

Therefore $\mathbf{h}^k \in S(\tilde{A}^*)$, $k = 1, 2$. As is easily checked

$$\mathbf{h} = (\mathbf{h}^1 + \mathbf{h}^2)/2,$$

and $\mathbf{h}^1 \neq \mathbf{h}^2$. Thus $\mathbf{h}$ is not an extreme point of $S(\tilde{T}^*)$.

($\Leftarrow$) Assume that (a) and (b) hold, and $\mathbf{h}$ is not an extreme point of $S(\tilde{\mathbf{A}}^*)$. Thus there exist $\mathbf{h}^1, \mathbf{h}^2 \in S(\tilde{A}^*)$ $\mathbf{h}^1 \neq \mathbf{h}^2$, such that

$$\mathbf{h} = (\mathbf{h}^1 + \mathbf{h}^2)/2.$$

Without loss of generality, we may assume that $\|\mathbf{h}\|_{A^*} = \|\mathbf{h}^1\|_{A^*} = \|\mathbf{h}^2\|_{A^*} = 1$. Set $c_i = \|h_i\|_{S_i^*}$, and $c_i^k = \|h_i^k\|_{S_i^*}$, $i = 1, \ldots, m$, $k = 1, 2$. Since

$$h_i = (h_i^1 + h_i^2)/2, \qquad i = 1, \ldots, m,$$

we have

$$c_i \leqslant (c_i^1 + c_i^2)/2, \qquad i = 1, ..., m.$$

From the definition of c_i, c_i^k and the respective norms, we also have

$$\|\mathbf{c}\|_{T^*} = \|\mathbf{c}^1\|_{T^*} = \|\mathbf{c}^2\|_{T^*} = 1.$$

Set

$$L = \{i : c_i = (c_i^1 + c_i^2)/2\}$$

and

$$M = \{i : c_i < (c_i^1 + c_i^2)/2\}.$$

From the definition of the c_i^k and c_i, it follows that if $i \in M$, then $c_i^1, c_i^2 > 0$. For each $i \in M$, let $\mu_i^1, \mu_i^2 \in [0, 1]$ satisfy

$$c_i = (\mu_i^1 c_i^1 + \mu_i^2 c_i^2)/2.$$

For $i \in L$, let $\mu_i^1 = \mu_i^2 = 1$. Set

$$\mathbf{c}_\mu^k = (\mu_1^k c_1^k, ..., \mu_m^k c_m^k), \qquad k = 1, 2.$$

Thus

$$\mathbf{c} = (\mathbf{c}_\mu^1 + \mathbf{c}_\mu^2)/2.$$

Since $\tilde{T}^*$ is a Banach lattice on $\mathbb{R}^m$ with respect to the usual elementwise order,

$$\|\mathbf{c}_\mu^k\|_{T^*} \leqslant \|\mathbf{c}^k\|_{T^*} = 1.$$

From (b), $\mathbf{c}$ is an extremal point of $S(\tilde{T}^*)$. Therefore

$$\mathbf{c} = \mathbf{c}_\mu^1 = \mathbf{c}_\mu^2.$$

Thus for $i \in L$, we necessarily have

$$c_i = c_i^1 = c_i^2.$$

If $i \in M$, we have $c_i^1, c_i^2 > 0$, and the above equality is only possible for all possible μ_i^k, $k = 1, 2$, if $c_i = 0$. That is, $M \subseteq \{i : c_i = 0\}$.

For $i \in L$ we have

$$h_i = (h_i^1 + h_i^2)/2$$

and $\|h_i\|_{S_i} = \|h_i^1\|_{S_i} = \|h_i^2\|_{S_i}$. Thus from (a), $h_i = h_i^1 = h_i^2$.

For $i \in M$, we have $h_i = 0$, and $h_i^2 = -h_i^1 \neq 0$. Therefore $\mathbf{c}^1 = \mathbf{c}^2 = \mathbf{b}$ where $b_i = c_i \geq 0$ for $i \in L$, and $b_i > c_i = 0$ for $i \in M$. Set

$$d_i = \begin{cases} b_i, & i \in L \\ -b_i, & i \in M. \end{cases}$$

By our assumption on $\tilde{T}$ (and hence on $\tilde{T}^*$),

$$\|\mathbf{d}\|_{T^*} = \|\mathbf{b}\|_{T^*} = 1.$$

Furthermore

$$\mathbf{c} = (\mathbf{b} + \mathbf{d})/2$$

and $\mathbf{b} \neq \mathbf{d}$ if $M \neq \varnothing$. This contradicts (b). As such $M = \varnothing$, and therefore

$$h_i = h_i^1 = h_i^2, \qquad i = 1, ..., m,$$

contradicting our hypothesis that $\mathbf{h}$ is not an extreme point for $S(\tilde{A}^*)$. ∎

The simple identification of the dual space of $A(p, q)$, and the extremal points thereof allows us to easily characterize best approximants. In addition, as a consequence of the above results, or via a more direct route, it is also possible to determine $\tau_+^{\tilde{A}}(\mathbf{f}, \mathbf{g})$, based on knowledge of $\tau_+^{S_i}$ and $\tau_+^{\tilde{T}}$. We have, see, e.g., Ioffe and Levin [13, p. 41],

$$\tau_+^{\tilde{A}}(\mathbf{f}, \mathbf{g}) = \tau_+^{\tilde{T}}(\{\|f_i\|_{S_i}\}_{i=1}^m, \{\tau_+^{S_i}(f_i, g_i)\}_{i=1}^m).$$

Finally, if $1 < p < \infty$ and $1 < q < \infty$, then $A(p, q)$ is a strictly convex normed linear space. As such, given a finite dimensional subspace of $A(p, q)$ we will always have uniqueness of the best approximant. We deal only with the remaining cases.

5. $A(1, q)$, $1 < q < \infty$

As in many of the cases to be considered, we will try to concentrate on the essential features of the problem. As such, let Y_i be a normed linear space with norm $\|\cdot\|_{Y_i}$, $i = 1, ..., m$. For the moment we assume that each Y_i is *smooth*. For each $f \in Y_i$, $f \neq 0$, we let $h_f \in Y_i^*$ denote the unique linear functional satisfying $\|h_f\|_{Y_i^*} = 1$, and $h_f(f) = \|f\|_{Y_i}$.

By Y we mean the normed linear space

$$Y = \{\mathbf{f} = (f_1, ..., f_m) : f_i \in Y_i, i = 1, ..., m\}$$

with norm

$$\|\mathbf{f}\|_Y = \sum_{i=1}^{m} \|f_i\|_{Y_i}.$$

If Y^* is the dual space to Y, then

$$Y^* = \{\mathbf{h} = (h_1, ..., h_m) : h_i \in Y_i^*, i = 1, ..., m\}$$

with norm

$$\|\mathbf{h}\|_{Y^*} = \max_{i=1,...,m} \|h_i\|_{Y_i^*}$$

Note that

$$\begin{aligned}\mathbf{h}(\mathbf{f}) &= \sum_{i=1}^{m} h_i(f_i) \leqslant \sum_{i=1}^{m} \|f_i\|_{Y_i} \|h_i\|_{Y_i^*} \\ &\leqslant \left(\sum_{i=1}^{m} \|f_i\|_{Y_i}\right) \left(\max_{i=1,...,m} \|h_i\|_{Y_i^*}\right) = \|\mathbf{f}\|_Y \|\mathbf{h}\|_{Y^*}.\end{aligned}$$

If $\mathbf{f} \in Y$, $\mathbf{f} \neq 0$, and $\mathbf{h} \in Y^*$ satisfies

$$\|\mathbf{h}\|_{Y^*} = 1, \qquad \mathbf{h}(\mathbf{f}) = \|\mathbf{f}\|_Y$$

then

(1) if $f_i \neq 0$, then $h_i = h_{f_i}$

(2) if $f_i = 0$, then $h_i \in Y_i^*$ may be arbitrarily chosen satisfying $\|h_i\|_{Y_i^*} \leqslant 1$.

As a general result, we have

THEOREM 5.1. *Let U be a finite dimensional subspace of Y. Then $\mathbf{u}^*$ is a best approximant to $\mathbf{f}$ from U if and only if*

$$\left| \sum_{\{i : f_i - u_i^* \neq 0\}} h_{f_i - u_i^*}(u_i) \right| \leqslant \sum_{\{i : f_i - u_i^* = 0\}} \|u_i\|_{Y_i} \tag{5.1}$$

for all $\mathbf{u} \in U$, or, equivalently, for i such that $f_i - u_i^* = 0$ there exist $h_i \in Y_i^*$, $\|h_i\|_{Y_i^*} \leqslant 1$, satisfying

$$\sum_{\{i : f_i - u_i^* \neq 0\}} h_{f_i - u_i^*}(u_i) + \sum_{\{i : f_i - u_i^* = 0\}} h_i(u_i) = 0 \tag{5.2}$$

for all $\mathbf{u} \in U$.

Proof. Assume $\mathbf{u}^*$ is a best approximant to $\mathbf{f}$ from U, $\mathbf{f} \neq \mathbf{u}^*$. From Theorem 2.1 there exists an $\mathbf{h} \in Y^*$ satisfying

(1) $\|\mathbf{h}\|_{Y^*} = 1$

(2) $\mathbf{h}(\mathbf{u}) = 0$, for all $\mathbf{u} \in U$

(3) $\mathbf{h}(\mathbf{f} - \mathbf{u}^*) = \|\mathbf{f} - \mathbf{u}^*\|_Y$.

Conditions (1) and (3) imply that $h_i = h_{f_i - u_i^*}$ if $f_i - u_i^* \neq 0$. Equation (5.2) is just a restatement of (2) where $\|\mathbf{h}\|_{Y^*} = 1$ implies that $\|h_i\|_{Y_i^*} \leq 1$.

If (5.2) holds, then for every $\mathbf{u} \in U$,

$$\left| \sum_{\{i : f_i - u_i^* \neq 0\}} h_{f_i - u_i^*}(u_i) \right| = \left| \sum_{\{i : f_i - u_i^* = 0\}} h_i(u_i) \right|$$

$$\leq \sum_{\{i : f_i - u_i^* = 0\}} \|u_i\|_{Y_i} \|h_i\|_{Y_i^*} \leq \sum_{\{i : f_i - u_i^* = 0\}} \|u_i\|_{Y_i}.$$

Thus (5.1) holds.

If (5.1) holds, then for any $\mathbf{u} \in U$,

$$\|\mathbf{f} - \mathbf{u}^*\|_Y = \sum_{i=1}^{m} \|f_i - u_i^*\|_{Y_i} = \sum_{\{i : f_i - u_i^* \neq 0\}} h_{f_i - u_i^*}(f_i - u_i^*)$$

$$= \sum_{\{i : f_i - u_i^* \neq 0\}} h_{f_i - u_i^*}(f_i - u_i) + \sum_{\{i : f_i - u_i^* \neq 0\}} h_{f_i - u_i^*}(u_i - u_i^*)$$

$$\leq \sum_{\{i : f_i - u_i^* \neq 0\}} \|f_i - u_i\|_{Y_i} \|h_{f_i - u_i^*}\|_{Y_i^*} + \sum_{\{i : f_i - u_i^* = 0\}} \|u_i - u_i^*\|_{Y_i}$$

$$= \sum_{\{i : f_i - u_i^* \neq 0\}} \|f_i - u_i\|_{Y_i} + \sum_{\{i : f_i - u_i^* = 0\}} \|u_i - f_i\|_{Y_i}$$

$$= \|\mathbf{f} - \mathbf{u}\|_Y.$$

Thus $\mathbf{u}^*$ is a best approximant of $\mathbf{f}$ from U. ∎

Remark. The fact that (5.2) implies (5.1) is trivial, as we have just seen. We did not directly prove that (5.1) implies (5.2), but rather proved it indirectly via the best approximation property. It may be directly proven that (5.1) implies (5.2) from general principles. This is a special case of what is sometimes called the abstract L-problem in normed linear spaces, see, e.g., Krein and Nudel'man [17, Chap. IX].

The above characterization result helps in determining when U is a unicity space.

PROPOSITION 5.2. *Assume there exsts a* $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$, *and* $\varepsilon_i \in \{-1, 1\}$, $i = 1, ..., m$, *such that*

$$\left| \sum_{\{i : u_i^* \neq 0\}} \varepsilon_i h_{u_i^*}(u_i) \right| \leqslant \sum_{\{i : u_i^* = 0\}} \|u_i\|_{Y_i} \tag{5.3}$$

for all $\mathbf{u} \in U$. *Then* U *is not a unicity space.*

Remark. We could also replace (5.3) by

$$\sum_{\{i : u_i^* \neq 0\}} \varepsilon_i h_{u_i^*}(u_i) + \sum_{\{i : u_i^* = 0\}} h_i(u_i) = 0 \tag{5.4}$$

for all $\mathbf{u} \in U$ and some $h_i \in Y_i^*$ satisfying $\|h_i\|_{Y_i^*} \leqslant 1$, for those i for which $u_i^* = 0$.

Proof. Let $f_i = \varepsilon_i u_i^*$, $i = 1, ..., m$. For $\alpha \in (-1, 1)$,

$$f_i - \alpha u_i^* = (\varepsilon_i - \alpha) u_i^*.$$

Since $1 = |\varepsilon_i| > |\alpha|$, it is easily seen that $f_i - \alpha u_i^* \neq 0$ if and only if $u_i^* \neq 0$, and if $f_i - \alpha u_i^* \neq 0$, then $h_{f_i - \alpha u_i^*} = \varepsilon_i h_{u_i^*}$. Thus (5.3) may be rewritten as

$$\left| \sum_{\{i : f_i - \alpha u_i^* \neq 0\}} h_{f_i - \alpha u_i^*}(u_i) \right| \leqslant \sum_{\{i : f_i - \alpha u_i^* = 0\}} \|u_i\|_{Y_i}.$$

From (5.1), this implies that $\alpha \mathbf{u}^*$ is a best approximant to $\mathbf{f}$ from U. Thus U is not a unicity space. ∎

For the converse result we impose an additional condition on the Y_i. We assume that each of the norms $\|\cdot\|_{Y_i}$ is *strictly convex*. Recall that in a strictly convex normed linear space, if

$$\|f + g\| = \|f\| + \|g\|,$$

then $f = 0$, $g = 0$, or $f = cg$ for some $c > 0$.

THEOREM 5.3. *Let* Y_i *be smooth and strictly convex normed linear spaces for each* $i = 1, ..., m$. *The finite dimensional subspace* U *of* Y *is a unicity space if and only if there does not exist a* $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ *and* $\varepsilon_i \in \{-1, 1\}$, $i = 1, ..., m$, *such that* (5.3) *holds for all* $\mathbf{u} \in U$.

Proof. ($\Rightarrow$) This is the content of the previous proposition.

($\Leftarrow$) Assume U is not a unicity space. Thus there exists an $\mathbf{f} \in Y$ and $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ such that $\pm \mathbf{u}^*$ are best approximants to $\mathbf{f}$ from U. Now,

$$2\|f_i\|_{Y_i} \leqslant \|f_i + u_i^*\|_{Y_i} + \|f_i - u_i^*\|_{Y_i}$$

for $i = 1, \ldots, m$. Furthermore,

$$2\,\|\mathbf{f}\|_Y = \|\mathbf{f} + \mathbf{u}^*\|_Y + \|\mathbf{f} - \mathbf{u}^*\|_Y.$$

Thus

$$2\,\|f_i\|_{Y_i} = \|f_i + u_i^*\|_{Y_i} + \|f_i - u_i^*\|_{Y_i}$$

for each $i = 1, \ldots, m$. Since each Y_i is strictly convex, this equality implies either $u_i^* = 0$ or $f_i = c_i u_i^*$, for some $|c_i| \geqslant 1$, for each i. Since $\mathbf{0}$ is a best approximant to $\mathbf{f}$ from U,

$$\left| \sum_{\{i : f_i \neq 0\}} h_{f_i}(u_i) \right| \leqslant \sum_{\{i : f_i = 0\}} \|u_i\|_{Y_i}$$

for all $\mathbf{u} \in U$. Now, if $u_i^* \neq 0$, then $f_i \neq 0$, and from the above,

$$h_{f_i} = \varepsilon_i h_{u_i^*},$$

where $\varepsilon_i = \operatorname{sgn} c_i$. If $f_i = 0$, then of course $u_i^* = 0$. Thus

$$\left| \sum_{\{i : u_i^* \neq 0\}} \varepsilon_i h_{u_i^*}(u_i) + \sum_{\{i : u_i^* = 0,\, f_i \neq 0\}} h_{f_i}(u_i) \right| \leqslant \sum_{\{i : f_i = 0\}} \|u_i\|_{Y_i}$$

for all $\mathbf{u} \in U$, which immediately implies (since $\|h_{f_i}\|_{Y_i} = 1$), that

$$\left| \sum_{\{i : u_i^* \neq 0\}} \varepsilon_i h_{u_i^*}(u_i) \right| \leqslant \sum_{\{i : u_i^* = 0\}} \|u_i\|_{Y_i}$$

for all $\mathbf{u} \in U$. That is, (5.3) holds. ∎

Let us consider some simple examples where (5.3) is, in a sense, easily checked.

EXAMPLE 1. $\dim U = 1$. In this case (5.3) reduces to the existence of $\varepsilon_i \in \{-1, 1\}$, $i = 1, \ldots, m$, such that

$$\sum_{\{i : u_i^* \neq 0\}} \varepsilon_i \|u_i^*\|_{Y_i} = 0.$$

Thus, under the assumptions of Theorem 5.3, $U = \operatorname{span}\{\mathbf{u}^*\}$ is a unicity space if and only if there do not exist $\varepsilon_i \in \{-1, 1\}$ satisfying

$$\sum_{i=1}^{m} \varepsilon_i \|u_i^*\|_{Y_i} = 0.$$

EXAMPLE 2. If for each $\mathbf{u} \in U \setminus \{\mathbf{0}\}$ there do *not* exist $\varepsilon_i \in \{-1, 1\}$, $i = 1, \ldots, m$, such that

$$\sum_{i=1}^{m} \varepsilon_i \|u_i\|_{Y_i} = 0,$$

then U is necessarily a unicity space.

EXAMPLE 3. Simultaneous Approximation. Let $Y_i = \tilde{Y}$, $i = 1, \ldots, m$, and

$$U = \{\mathbf{u} : \mathbf{u} = (u, \ldots, u), u \in \tilde{U} \subset \tilde{Y}\}.$$

Thus $\dim U = \dim \tilde{U}$. We assume that $\tilde{Y}$ is smooth and strictly convex. Then U is a unicity space if and only if m is odd. For in this case of simultaneous approximation, if $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$, then $u_i^* = u^* \neq 0$ for every i, and (5.3) reduces to

$$\sum_{i=1}^{m} \varepsilon_i h_{u^*}(u) = 0$$

for all $u \in \tilde{U}$. That is,

$$h_{u^*}(u) \left(\sum_{i=1}^{m} \varepsilon_i \right) = 0$$

for all $u \in \tilde{U}$. If m is even, let $\varepsilon_i = (-1)^i$, $i = 1, \ldots, m$, and the above always holds. If m is odd,

$$\sum_{i=1}^{m} \varepsilon_i \neq 0$$

and

$$h_{u^*}(u) \neq 0$$

for some $u \in \tilde{U}$, e.g., $u = u^*$.

EXAMPLE 4. "Tensor" Product. Let $U = V \oplus W$, where for all $\mathbf{v} \in V$, $v_i = 0$, $i = l+1, \ldots, m$, while for all $\mathbf{w} \in W$, $w_i = 0$, $i = 1, \ldots, l$ $(1 \leqslant l < m)$. Then, as is easily checked, U is a unicity space if and only if both V and W are unicity spaces.

Looking back at the previous theorem and the method of proof thereof, it follows that we have proved this next result.

COROLLARY 5.4. *If* $\mathbf{f} \in Y$ *is such that* $f_i \neq u_i$ *for any* $i \in \{1, \ldots, m\}$, *and any* $\mathbf{u} \in U$, *then* $\mathbf{f}$ *has a unique best approximant from* U.

Inequality (5.3) is not a condition which is in general easy to check. As such, we take specific choices for the Y_i, $i = 1, \ldots, m$, and examine them in further detail.

Let K be a compact subset of $\mathbb{R}^d$, $K = \overline{\operatorname{int} K}$. Let μ_i be non-negative σ-finite measures on K. We set $Y_i = L^{q_i}(K, \mu_i)$, where $1 < q_i < \infty$, $i = 1, \ldots, m$. Thus each Y_i is smooth and strictly convex. We have that $f \in L^{q_i}(K, \mu_i)$ if

$$\|f\|_{q_i} = \left(\int_K |f(x)|^{q_i} \, d\mu_i(x) \right)^{1/q_i}$$

exists and is finite. We can restate Theorem 5.3 as follows: U is a unicity space if and only if there does not exist a $\mathbf{u}^* \in U \backslash \{\mathbf{0}\}$, and $\varepsilon_i \in \{-1, 1\}$, $i = 1, \ldots, m$, such that

$$\left| \sum_{\{i : u_i^* \neq 0\}} \varepsilon_i \int_K \frac{|u_i^*(x)|^{q_i - 1} \operatorname{sgn}(u_i^*(x)) \, u_i(x)}{\|u_i^*\|_{q_i}^{q_i - 1}} \, d\mu_i(x) \right| \leqslant \sum_{\{i : u_i^* = 0\}} \|u_i\|_{q_i}$$

for all $\mathbf{u} \in U$.

This is certainly a condition which is not easy to verify and depends on the specific $\{\mu_i\}$. We look for a condition which is essentially $\{\mu_i\}$ independent. To simplify matters we restrict ourselves to measures $\mu_i \in \mathscr{A}$ (see Section 3). Under these assumptions, we have:

THEOREM 5.5. *U is a unicity space for all choices of $\{\mu_i\}_{i=1}^m \in \mathscr{A}$ if and only if for each $\mathbf{u}^* \in U \backslash \{\mathbf{0}\}$ and $\varepsilon_i \in \{-1, 1\}$, $i = 1, \ldots, m$, there exists a $\mathbf{v} \in U$ satisfying*

$$\begin{aligned} &\text{(a)} \quad v_i = 0 \textit{ if } u_i^* = 0 \\ &\text{(b)} \quad \varepsilon_i u_i^*(x) \, v_i(x) \geqslant 0 \textit{ (Lebesgue) a.e. on } K,\ i = 1, \ldots, m \\ &\text{(c)} \quad \varepsilon_j u_j^*(x) \, v_j(x) > 0 \textit{ for some } j \in \{1, \ldots, m\} \textit{ on a} \\ &\qquad \textit{set of positive Lebesgue measure.} \end{aligned} \tag{5.5}$$

Proof. ($\Leftarrow$) Assume that given any $\mathbf{u}^* \in U \backslash \{\mathbf{0}\}$ and $\varepsilon_i \in \{-1, 1\}$, $i = 1, \ldots, m$, there exists a $\mathbf{v} \in U$ satisfying (5.5). Assume U is not a unicity space for some measures $\{\mu_i\}_{i=1}^m \in \mathscr{A}$. There then exists a $\mathbf{u}^* \in U \backslash \{\mathbf{0}\}$ and $\varepsilon_i \in \{-1, 1\}$, $i = 1, \ldots, m$, such that

$$\left| \sum_{\{i : u_i^* \neq 0\}} \varepsilon_i \int_K \frac{|u_i^*(x)|^{q_i - 1} \operatorname{sgn}(u_i^*(u)) \, u_i(x)}{\|u_i^*\|_{q_i}^{q_i - 1}} \, d\mu_i(x) \right| \leqslant \sum_{\{i : u_i^* = 0\}} \|u_i\|_{q_i},$$

for all $\mathbf{u} \in U$. Let $\mathbf{v} \in U$ satisfy (5.5) with respect to this $\mathbf{u}^*$ and these $\{\varepsilon_i\}_{i=1}^m$. From (a)

$$\sum_{\{i : u_i^* \neq 0\}} \varepsilon_i \int_K \frac{|u_i^*(x)|^{q_i - 1} \operatorname{sgn}(u_i^*(x))\, v_i(x)}{\|u_i^*\|_{q_i}^{q_i - 1}}\, d\mu_i(x) = 0.$$

From (b) and (c), it follows that the above quantity is strictly positive. This contradiction implies that U is a unicity space.

($\Rightarrow$) Assume there exists a $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ and $\varepsilon_i \in \{-1, 1\}$, $i = 1, ..., m$, so that no $\mathbf{v} \in U$, $\mathbf{v} \neq \mathbf{0}$, satisfies (5.5). That is, in particular, if $\mathbf{v} \in U$ satisfies

(a′) $v_i = 0$ if $u_i^* = 0$

(b′) $\varepsilon_i u_i^*(x)\, v_i(x) \geqslant 0$ μ a.e. on K, $i = 1, ..., m$

(where $d\mu$ is Lebegue measure), then $u_i^* v_i = 0$ μ a.e. for each $i = 1, ..., m$.

Set

$$U_1 = \{\mathbf{u} : \mathbf{u} \in U, u_i = 0 \text{ if } u_i^* = 0\}.$$

The set U_1 is a linear subspace of U of dimension k, $1 \leqslant k \leqslant n$ (since $\mathbf{u}^* \in U_1$).

Let

$$U_1 = \operatorname{span}\{\mathbf{u}^1, ..., \mathbf{u}^k\}.$$

Choose $\mathbf{u}^{k+1}, ..., \mathbf{u}^n$ so that

$$U = \operatorname{span}\{\mathbf{u}^1, ..., \mathbf{u}^k, \mathbf{u}^{k+1}, ..., \mathbf{u}^n\}$$

and set

$$U_2 = \operatorname{span}\{\mathbf{u}^{k+1}, ..., \mathbf{u}^n\}.$$

Thus $U = U_1 \oplus U_2$. Let $Q = \{i : u_i^* = 0\}$, and $P = \{1, ..., m\} \setminus Q$. Note that the $\mathbf{u}^{k+1}, ..., \mathbf{u}^n$ are "linearly independent over Q." That is, if $\mathbf{u} \in U_2$ satisfies $u_i = 0$ for all $i \in Q$, then $\mathbf{u} = \mathbf{0}$.

For $i \in Q$, let $\tilde{h}_i \in L^\infty(K)$, satisfy

(i) $|\tilde{h}_i(x)| = 1$, for all $x \in K$

(ii) $\int_K \tilde{h}_i(x)\, u_i(x)\, d\mu(x) = 0$ for all $\mathbf{u} \in U$.

Such $\tilde{h}_i$ exist since U is of finite dimension, and by our various assumptions.

Let

$$\mathbf{h}^* = (h_1^*, ..., h_m^*),$$

where for $i \in Q$, $h_i^* = \tilde{h}_i$, and for $i \in P$, $h_i^* = \varepsilon_i\, |u_i^*(x)|^{q_i - 1} \operatorname{sgn}(u_i^*(x))$.

Set

$$W = \{\mathbf{w} : \mathbf{w} = \mathbf{h}^* \cdot \mathbf{u},\ \mathbf{u} \in U\},$$

where by $\mathbf{w} = \mathbf{h}^* \cdot \mathbf{u}$, we mean that $w_i(x) = h_i^*(x)\, u_i(x)$, $i = 1, ..., m$. Note that W is a linear subspace of dimension at most n.

We claim that W contains no non-negative nontrivial function. Assume that $\mathbf{w} \in W$ satisfies $w_i \geq 0$ μ a.e., $i = 1, ..., m$. Now $w_i = h_i^* u_i = h_i^*(v_i^1 + v_i^2)$, where $\mathbf{v}^j \in U_j$, $j = 1, 2$. For $i \in Q$, $v_i^1 = 0$ and $h_i^* = \tilde{h}_i$. Thus

$$w_i = \tilde{h}_i\, v_i^2 \geq 0$$

μ a.e. for $i \in Q$. But from (ii), we see that $\tilde{h}_i\, v_i^2 = 0$ μ a.e., and using (i), we have $v_i^2 = 0$ μ a.e. for such i. By our definition of U_2, this implies that $\mathbf{v}^2 = \mathbf{0}$. Thus

$$\mathbf{w} = \mathbf{h}^* \cdot \mathbf{v}^1.$$

Let us rename $\mathbf{v}^1$ as $\mathbf{v}$. Since $\mathbf{v} \in U^1$ we have that $v_i = 0$ for all $i \in Q$, i.e., (a′) holds. Now $h_i^* v_i \geq 0$ for $i \in P$ translates into

$$\varepsilon_i\, |u_i^*(x)|^{q_i - 1}\, \mathrm{sgn}(u_i^*(x))\, v_i(x) \geq 0$$

μ a.e., which is equivalent to (b′). If $\mathbf{w}$ is non-negative and non-trivial then $u_j^* v_j > 0$ for some $j \in P$ on some set of positive measure. But this contradicts our assumption.

Since W contains no non-negative non-trivial function (and $\dim W < \infty$), there exist measures $\{\mu_i\}_{i=1}^m$ in $\mathscr{A}$ such that

$$\sum_{i=1}^m \int_K w_i(x)\, d\mu_i(x) = 0$$

for all $\mathbf{w} \in W$ (see, e.g., Pinkus [29, p. 61]). In fact we could choose the μ_i to be of the form $\sigma_i(x)\, dx$, where $\sigma_i \in C(K)$, $\sigma_i > 0$. Thus

$$\sum_{i=1}^m \int_K h_i^*(x)\, u_i(x)\, d\mu_i(x) = 0$$

for all $\mathbf{u} \in U$. That is,

$$\sum_{i \in P} \varepsilon_i \int_K |u_i^*(x)|^{q_i - 1}\, \mathrm{sgn}(u_i^*(x))\, u_i(x)\, d\mu_i(x)$$

$$+ \sum_{i \in Q} \int_K \tilde{h}_i(x)\, u_i(x)\, d\mu_i(x) = 0$$

for all $\mathbf{u} \in U$.

Set $\tilde{\mu}_i = c_i \mu_i$, $i = 1, ..., m$, where for $i \in P$

$$c_i = \left(\int_K |u_i^*(x)|^{q_i} \, d\mu_i \right)^{q_i - 1},$$

and for $i \in Q$

$$c_i \geqslant (\mu_i(K))^{q_i - 1}.$$

Thus for each $\mathbf{u} \in U$,

$$\sum_{i \in P} \varepsilon_i \int_K \frac{|u_i^*(x)|^{q_i - 1} \operatorname{sgn}(u_i^*(x)) \, u_i(x)}{c_i} \, d\tilde{\mu}_i(x) + \sum_{i \in Q} \int_K \frac{\tilde{h}_i(x)}{c_i} u_i(x) \, d\tilde{\mu}_i(x) = 0.$$

For $i \in P$,

$$\begin{aligned}
\|u_i^*\|_{q_i}^{q_i - 1} &= \left(\int_K |u_i^*(x)|^{q_i} \, d\tilde{\mu}_i(x) \right)^{(q_i - 1)/q_i} \\
&= \left(\int_K |u_i^*(x)|^{q_i} c_i \, d\mu_i(x) \right)^{(q_i - 1)/q_i} \\
&= c_i^{(q_i - 1)/q_i} \left(\int_K |u_i^*(x)|^{q_i} \, d\mu_i(x) \right)^{(q_i - 1)/q_i} \\
&= c_i^{1 - (1/q_i)} c_i^{1/q_i} \\
&= c_i.
\end{aligned}$$

For $i \in Q$,

$$\left| \int_K \frac{\tilde{h}_i(x)}{c_i} u_i(x) \, d\tilde{\mu}_i(x) \right| \leqslant \left\| \frac{\tilde{h}_i}{c_i} \right\|_{q_i'} \|u_i\|_{q_i}.$$

A simple calculation shows that

$$\left\| \frac{\tilde{h}_i}{c_i} \right\|_{q_i'} \leqslant 1.$$

Substituting we obtain

$$\left| \sum_{\{i : u_i^* \neq 0\}} \varepsilon_i \int_K \frac{|u_i^*(x)|^{q_i - 1} \operatorname{sgn}(u_i^*(x)) \, u_i(x)}{\|u_i^*\|_{q_i}^{q_i - 1}} \, d\tilde{\mu}_i(x) \right| \leqslant \sum_{\{i : u_i^* = 0\}} \|u_i\|_{q_i}$$

for all $\mathbf{u} \in U$.

Thus U is not a unicity space with respect to the measures $\tilde{\mu}_1, ..., \tilde{\mu}_m$. ∎

Theorem 5.5 is of course vacuous in the case where $m=1$. For if $m=1$, then every finite dimensional subspace is a unicity space since the norm is strictly convex. Thus we always assume $m \geqslant 2$. If U is a tensor product of the form

$$U = U^1 \oplus \cdots \oplus U^m,$$

where for each $\mathbf{u} \in U^j$ we have $u_i = 0$ for all $i \neq j$, then it is easily seen that U is a unicity space. For in this case,

$$\min_{\mathbf{u} \in U} \|\mathbf{f} - \mathbf{u}\| = \sum_{i=1}^{m} \min_{u_i \in \tilde{U}^i} \|f_i - u_i\|_{Y_i},$$

where $\tilde{U}^i = \{u_i : \mathbf{u} \in U^i\}$. Since each $\tilde{U}^i$ is a finite dimensional subspace of a strictly convex normed linear space Y_i, the problem

$$\min_{u_i \in \tilde{U}^i} \|f_i - u_i\|_{Y_i}$$

has a unique solution u_i^* and $\mathbf{u}^* = (u_1^*, ..., u_m^*)$ is the unique solution to our original minimization problem.

The question naturally arises as to which other subspaces U satisfy the conditions of Theorem 5.5. We conjecture that there are no others. For $n=1$, this is easily seen. For assume $U = \operatorname{span}\{\mathbf{u}^*\}$, and $u_i^* \neq 0$, $u_j^* \neq 0$, $i \neq j$. Let $\varepsilon_i \varepsilon_j = -1$, and $\varepsilon_k \in \{-1, 1\}$ for all other k. Then (5.5) cannot hold. We can prove this conjecture for $n \geqslant 2$ under somewhat restrictive assumptions.

THEOREM 5.6. *Assume $m \geqslant 2$, and U is a finite dimensional subspace for which*

(i) *$u_i \in C(K)$, $i = 1, ..., m$, all $\mathbf{u} \in U$.*

(ii) *If $u_i \neq 0$, then $\mu(Z(u_i)) = 0$ and $[K \setminus Z(u_i)] \leqslant M$ for some M independent of $\mathbf{u} \in U$ and i.*

Then U satisfies (5.5) *if and only if*

$$U = U^1 \oplus \cdots \oplus U^m$$

as above.

Remark. Recall that $Z(u_i) = \{x : u_i(x) = 0\}$, μ is Lebesgue measure, and $[K \setminus Z(u_i)]$ denotes the number of connected components in the set $K \setminus Z(u_i)$.

Proof. One direction is obvious. We therefore assume that U satisfies (5.5) of Theorem 5.5, and (i) and (ii), and prove that it is then necessarily a tensor product as above.

For $\mathbf{u} \in U$, set $P_i \mathbf{u} = u_i$, $i = 1, ..., m$. That is, P_i is the projection onto the "ith" component. Let

$$n_i = \dim P_i U = \dim \operatorname{span}\{u_i : \mathbf{u} \in U\}$$

and $n = \dim U$.

For any subspace U, $n \leqslant \sum_{i=1}^{m} n_i$. Our claim is equivalent to proving that $n = \sum_{i=1}^{m} n_i$. There is also a different equivalent form of our claim. For each $i \in \{1, ..., m\}$, set

$$Q_i \mathbf{u} = (u_1, ..., u_{i-1}, u_{i+1}, ..., u_m).$$

Let $m_i = \dim Q_i U$. Now $n_i + m_i \geqslant n$. Our claim is equivalent to proving that $n_i + m_i = n$ for each $i = 1, ..., m$.

Assume $n_i + m_i > n$ for some $i \in \{1, ..., m\}$. For ease of exposition, assume that $i = 1$. For each $\mathbf{u} \in U$, let $\tilde{\mathbf{u}} = Q_1 \mathbf{u} = (u_2, ..., u_m)$.

Since $n_1 + m_1 > n$, we have $n_1 + m_1 = n + l$, $l \geqslant 1$, and there exist

$$\mathbf{u}^1, ..., \mathbf{u}^l, \mathbf{v}^{l+1}, ..., \mathbf{v}^{n_1}, \mathbf{w}^{l+1}, ..., \mathbf{w}^{m_1},$$

a basis for U such that

(1) $u_1^1, ..., u_1^l, v_1^{l+1}, ..., v_1^{n_1}$ is a basis for $P_1 U$.

(2) $\tilde{\mathbf{u}}^1, ..., \tilde{\mathbf{u}}^l, \tilde{\mathbf{w}}^{l+1}, ..., \tilde{\mathbf{w}}^{m_1}$ is a basis for $Q_1 U$.

(3) $\tilde{\mathbf{v}}^j = \mathbf{0}$, $j = l+1, ..., n_1$.

(4) $w_1^j = 0$, $j = l+1, ..., m_1$.

Using Proposition 4.13 of Pinkus [29] and the assumptions (i) and (ii), there exists a

$$p_1^* = u_1^1 + \sum_{j=2}^{l} a_j^* u_1^j + \sum_{j=l+1}^{n_1} b_j^* v_1^j$$

such that if $p \in P_1 U$ and $p_1^* p \geqslant 0$, then $p = \alpha_1 p_1^*$ for some $\alpha_1 \geqslant 0$. Let

$$\mathbf{w}^* = \mathbf{u}^1 + \sum_{j=2}^{l} a_j^* \mathbf{u}^j,$$

and

$$W = \operatorname{span}\{\tilde{\mathbf{w}}^*, \tilde{\mathbf{w}}^{l+1}, ..., \tilde{\mathbf{w}}^{m_1}\}.$$

From (2), dim $W=m_1-l+1$. From Proposition 4.13 of Pinkus [29] and the assumptions (i) and (ii), there exists a

$$\tilde{\mathbf{p}}^* = \tilde{\mathbf{w}}^* + \sum_{j=l+1}^{m_1} c_j^* \tilde{\mathbf{w}}^j$$

such that if $\tilde{\mathbf{p}} \in W$, $\tilde{p}_i \tilde{p}_i^* \geqslant 0$, $i=2, ..., m$, and $\tilde{p}_i=0$ if $\tilde{p}_i^*=0$, then $\tilde{\mathbf{p}}=\alpha_2 \tilde{\mathbf{p}}^*$, $\alpha_2 \geqslant 0$.

Set

$$\mathbf{p}^* = \mathbf{u}^1 + \sum_{j=2}^{l} a_j^* \mathbf{u}^j + \sum_{j=l+1}^{n_1} b_j^* \mathbf{v}^j + \sum_{j=l+1}^{m_1} c_j^* \mathbf{w}^j.$$

Note that $P_1 \mathbf{p}^* = p_1^*$, and $Q_1 \mathbf{p}^* = \tilde{\mathbf{p}}^*$. Since U satisfies (5.5) (by assumption), there exists a $\mathbf{z}^* \in U \setminus \{\mathbf{0}\}$ such that

(a) $z_i^*=0$ if $p_i^*=0$.

(b_1) $z_1^*(x)\, p_1^*(x) \geqslant 0$, for all $x \in K$.

(b_2) $z_j^*(x)\, p_j^*(x) \leqslant 0$, for all $x \in K$, and $j=2, ..., m$.

(c) $\int_K z_1^* p_1^*\, d\mu - \sum_{j=2}^{m} \int_K z_j^* p_j^*\, d\mu > 0$ for μ Lebesgue measure.

Let

$$\mathbf{z}^* = \sum_{j=1}^{l} a_j \mathbf{u}^j + \sum_{j=l+1}^{n_1} b_j \mathbf{v}^j + \sum_{j=l+1}^{m_1} c_j \mathbf{w}^j.$$

From (b_1) (i.e., $z_1^* p_1^* \geqslant 0$), we have

$$z_1^* = \alpha_1 p_1^*, \qquad \alpha_1 \geqslant 0.$$

Thus $a_j = \alpha_1 a_j^*$, $j=1, ..., l$ ($a_1^*=1$), and $b_j=\alpha_1 b_j^*$, $j=l+1, ..., n_1$. Thus

$$\mathbf{z}^* = \alpha_1 \left(\mathbf{w}^* + \sum_{j=l+1}^{n_1} b_j^* \mathbf{v}^j \right) + \sum_{j=l+1}^{m_1} c_j \mathbf{w}^j$$

and therefore

$$\tilde{\mathbf{z}}^* = \alpha_1 \tilde{\mathbf{w}}^* + \sum_{j=l+1}^{m_1} c_j \tilde{\mathbf{w}}^j \in W.$$

From (a) and (b_2),

$$\tilde{\mathbf{z}}^* = \alpha_2 \tilde{\mathbf{p}}^*, \qquad \alpha_2 \leqslant 0,$$

i.e., $\tilde{\mathbf{z}}^* = \alpha_2 (\tilde{\mathbf{w}}^* + \sum_{j=l+1}^{m_1} c_j^* \tilde{\mathbf{w}}^j)$.

Since the $\{\tilde{\mathbf{w}}, \tilde{\mathbf{w}}^{l+1}, ..., \tilde{\mathbf{w}}^{m_1}\}$ are linearly independent, we get $\alpha_1 = \alpha_2$. But $\alpha_1 \geqslant 0 \geqslant \alpha_2$. Thus $\alpha_1 = \alpha_2 = 0$, which implies that $a_j = 0$, $j = 1, ..., l$; $b_j = 0$, $j = l+1, ..., n_1$; and $c_j = 0$, $j = l+1, ..., m_1$. That is, $\mathbf{z}^* = \mathbf{0}$. This contradicts (c). Thus $n_1 + m_1 = n$. ∎

6. $A(p, 1)$, $1 < p < \infty$

Let D_i be a set, Σ_i a σ-field of subsets of D_i, and ν_i a positive σ-finite measure defined on Σ_i, $i = 1, ..., m$. For each i, we let $L^1(D_i, \nu_i)$ $(= L^1(\nu_i))$ denote the Banach space of ν_i-measurable functions f_i defined on D_i for which $|f_i|$ is ν_i-integrable over D_i, with norm

$$\|f_i\|_{L^1(\nu_i)} = \int_{D_i} |f_i(x)|\, d\nu_i(x), \qquad i = 1, ..., m.$$

For $1 < p < \infty$, let Y_p denote the Banach space

$$Y_p = \{\mathbf{f} = (f_1, ..., f_m) : f_i \in L^1(D_i, \nu_i), i = 1, ..., m\}$$

with norm

$$\|\mathbf{f}\|_{Y_p} = \left(\sum_{i=1}^{m} \|f_i\|_{L^1(\nu_i)}^p \right)^{1/p}.$$

We herein assume, for convenience only, that $D_i = D$, $i = 1, ..., m$. The dual space Y_p^* of Y_p may be identified with

$$Y_p^* = \{\mathbf{h} = (h_1, ..., h_m) : h_i \in L^\infty(D, \nu_i), i = 1, ..., m\}$$

with norm

$$\|\mathbf{h}\|_{Y_p^*} = \left(\sum_{i=1}^{m} \|h_i\|_{L^\infty(\nu_i)}^{p'} \right)^{1/p'},$$

where $1/p + 1/p' = 1$. From the fact that

$$\begin{aligned}
\mathbf{h}(\mathbf{f}) &= \sum_{i=1}^{m} h_i(f_i) = \sum_{i=1}^{m} \int_D f_i(x)\, h_i(x)\, d\nu_i(x) \\
&\leqslant \sum_{i=1}^{m} \|f_i\|_{L^1(\nu_i)} \|h_i\|_{L^\infty(\nu_i)} \\
&\leqslant \left(\sum_{i=1}^{m} \|f_i\|_{L^1(\nu_i)}^p \right)^{1/p} \left(\sum_{i=1}^{m} \|h_i\|_{L^\infty(\nu_i)}^{p'} \right)^{1/p'}
\end{aligned}$$

we have that if $\mathbf{f} \in Y_p$, $\mathbf{f} \neq \mathbf{0}$, and $\mathbf{h} \in Y_p^*$ satisfies $\|\mathbf{h}\|_{Y_p^*} = 1$ and

$$\mathbf{h}(\mathbf{f}) = \|\mathbf{f}\|_{Y_p},$$

then equality holds in both above inequalities. As such,

(1)(a) $h_i(x) = c_i \operatorname{sgn}(f_i(x))$, v_i a.e. on $N(f_i)$

(b) $\|h_i\|_{L^\infty(v_i)} = c_i$

(2) $c_i = (\|f_i\|_{L^1(v_i)} / \|\mathbf{f}\|_{Y_p})^{p-1}$

for each $i = 1, \ldots, m$.

Characterization of best approximants is easily obtained from an application of Theorem 2.1.

THEOREM 6.1. *Assume U is a finite dimensional subspace of Y_p. Then $\mathbf{u}^*$ is a best approximant to $\mathbf{f}$ from U if and only if there exist $h_i \in L^\infty(Z(f_i - u_i^*), v_i)$ satisfying $\|h_i\|_{L^\infty(v_i)} \leqslant 1$ and*

$$0 = \sum_{i=1}^{m} \|f_i - u_i^*\|_{L^1(v_i)}^{p-1} \left[\int_{N(f_i - u_i^*)} (\operatorname{sgn}(f_i - u_i^*)(x))\, u_i(x)\, dv_i(x) \right.$$
$$\left. + \int_{Z(f_i - u_i^*)} h_i(x)\, u_i(x)\, dv_i(x) \right] \tag{6.1}$$

for all $\mathbf{u} \in U$. Or, equivalently to (6.1), we have

$$\left| \sum_{i=1}^{m} \|f_i - u_i^*\|_{L^1(v_i)}^{p-1} \int_K (\operatorname{sgn}(f_i - u_i^*)(x))\, u_i(x)\, dv_i(x) \right|$$
$$\leqslant \sum_{i=1}^{m} \|f_i - u_i^*\|_{L^1(v_i)}^{p-1} \int_{Z(f_i - u_i^*)} |u_i(x)|\, dv_i(x) \tag{6.2}$$

for all $\mathbf{u} \in U$.

In what follows, we assume that each v_i is non-atomic (as well as satisfying the previous assumptions). The fact that the measures are non-atomic allows us to assume in (6.1) that $|h_i(x)| = 1$ for all $x \in Z(f_i - u_i^*)$. This is a consesequence of the Liapounoff Theorem (see Liapounoff [23]), which we use in the proof of this generalization of Theorem 3.5.

PROPOSITION 6.2. *If each v_i is non-atomic (and as above), $i = 1, \ldots, m$, then no finite dimensional subspace U of Y_p is a unicity space.*

Proof. Choose $h_i^* \in L^\infty(v_i)$ to satisfy

(1) $|h_i^*(x)| = 1$, all $x \in D$

(2) $\int_D h_i^*(x)\, u_i(x)\, dv_i(x) = 0$ all $\mathbf{u} \in U$, and all $i = 1, \ldots, m$.

Such h_i^* necessarily exist (as a consequence of the Liapounoff Theorem) since the ν_i are non-atomic and

$$P_i U = \{u_i : \mathbf{u} \in U\}$$

is finite dimensional.

Let $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ be arbitrarily chosen. Set

$$f_i^*(x) = h_i^*(x)\, |u_i^*(x)|, \qquad i = 1, ..., m,$$

and $\mathbf{f}^* = (f_1^*, ..., f_m^*)$. Note that $N(f_i^*) = N(u_i^*)$, and $\operatorname{sgn}(f_i^*(x)) = h_i^*(x)$ on $N(f_i^*)$ for each $i = 1, ..., m$. Thus,

$$\begin{aligned}
&\sum_{i=1}^{m} \|f_i^*\|_{L^1(\nu_i)}^{p-1} \left[\int_{N(f_i^*)} (\operatorname{sgn}(f_i^*(x)))\, u_i(x)\, d\nu_i(x) \right. \\
&\qquad \left. + \int_{Z(f_i^*)} h_i^*(x)\, u_i(x)\, d\nu_i(x) \right] \qquad (6.3) \\
&= \sum_{i=1}^{m} \|f_i^*\|_{L^1(\nu_i)}^{p-1} \int_D h_i^*(x)\, u_i(x)\, d\nu_i(x) \\
&= 0.
\end{aligned}$$

From Proposition 6.1, we have that $\mathbf{0}$ is a best approximant to $\mathbf{f}$ from U.

For $\alpha \in (-1, 1)$ and each $i \in \{1, ..., m\}$, it follows since $|h_i^*(x)| = 1$ for all x, that $N(f_i^* - \alpha u_i^*) = N(f_i^*) = N(u_i^*)$, and $\operatorname{sgn}((f_i^* - \alpha u_i^*)(x)) = \operatorname{sgn}(f_i^*(x)) = h_i^*(x)$ for all $x \in N(f_i^*)$. Furthermore,

$$\begin{aligned}
\|f_i^* - \alpha u_i^*\|_{L^1(\nu_i)} &= \int_{N(f_i^*)} \operatorname{sgn}(f_i^*(x))(f_i^* - \alpha u_i^*)(x)\, d\nu_i(x) \\
&= \|f_i^*\|_{L^1(\nu_i)} - \alpha \int_{N(u_i^*)} h_i^*(x)\, u_i^*(x)\, d\nu_i(x) \\
&= \|f_i^*\|_{L^1(\nu_i)} - \alpha \int_D h_i^*(x)\, u_i^*(x)\, d\nu_i(x) \\
&= \|f_i^*\|_{L^1(\nu_i)}.
\end{aligned}$$

Thus $\|\mathbf{f}\|_{Y_p} = \|\mathbf{f} - \alpha \mathbf{u}^*\|_{Y_p}$ and $\alpha \mathbf{u}^*$ is also a best approximant to $\mathbf{f}^*$ from U. U is not a unicity space. ∎

We now make some further additional assumptions. We replace D by K where $K \subset \mathbb{R}^d$, compact, $K = \overline{\operatorname{int} K}$, as in Section 3. For each $\mathbf{u} \in U$, $u_i \in C(K)$ and μ_i (replacing ν_i) is in $\mathscr{A}$. We also let CY_p denote the restriction of Y_p to $\mathbf{f} = (f_1, ..., f_m)$ such that $f_i \in C(K)$, $i = 1, ..., m$. As a generalization of Theorem 3.6, we have the following characterization of finite dimensional unicity spaces for CY_p which, it should be noted, is independent of $p \in (1, \infty)$.

THEOREM 6.3. *The finite dimensional subspace U of CY_p is a unicity space for CY_p if and only if there do not exist $h_i^* \in L^\infty(K, \mu_i)$, $i = 1, \ldots, m$, a $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$, and numbers $\{\lambda_i^*\}_{i=1}^m$ such that*

(1) $|h_i^*(x)| = 1$, *all* $x \in K$, $i = 1, \ldots, m$.

(2) $h_i^*(x)\,|u_i^*(x)| \in C(K)$, $i = 1, \ldots, m$.

(3) $\sum_{i=1}^m \lambda_i^* \int_K h_i^*(x)\, u_i(x)\, d\mu_i(x) = 0$ *for all* $\mathbf{u} \in U$, *where* $\lambda_i^* \geqslant 0$, *and* $\lambda_i^* > 0$ *if* $\|u_i^*\|_{L^1(\mu_i)} > 0$.

(4) $\int_K h_i^*(x)\, u_i^*(x)\, d\mu_i(x) = 0$, $i = 1, \ldots, m$.

Proof. ($\Rightarrow$) Assume there exist h_i^*, u_i^* and λ_i^*, $i = 1, \ldots, m$, satisfying (1), (2), (3), and (4).

If $\|u_i^*\|_{L^1(\mu_i)} > 0$, set

$$f_i(x) = \frac{\lambda_i^{*\,1/(p-1)}}{\|u_i^*\|_{L^1(\mu_i)}}\, h_i^*(x)\, |u_i^*(x)|.$$

Note that $f_i \in C(K)$ and $\|f_i\|_{L^1(\mu_i)} = \lambda_i^{*\,1/(p-1)}$. If $u_i^* = 0$ and $\lambda_i^* = 0$, set $f_i = 0$. If $u_i^* = 0$ and $\lambda_i^* > 0$, let f_i be any function in $C(K)$ satisfying

$$\|f_i\|_{L^1(\mu_i)} = \lambda_i^{*\,1/(p-1)}$$

and $\operatorname{sgn} f_i(x) = h_i^*(x)$ on $N(f_i)$.

Thus, for every $\mathbf{u} \in U$

$$\begin{aligned} &\sum_{i=1}^m \|f_i\|_{L^1(\mu_i)}^{p-1} \left[\int_{N(f_i)} (\operatorname{sgn} f_i(x))\, u_i(x)\, d\mu_i(x) + \int_{Z(f_i)} h_i^*(x)\, u_i(x)\, d\mu_i(x) \right] \\ &\quad = \sum_{i=1}^m \lambda_i^* \int_K h_i^*(x)\, u_i(x)\, d\mu_i(x) \\ &\quad = 0 \end{aligned}$$

by (3). Thus $\mathbf{0}$ is a best approximant to $\mathbf{f}$ from U.

Let α satisfy

$$|\alpha| < \min \left\{ \frac{\lambda_i^{*\,1/(p-1)}}{\|u_i^*\|_{L^1(\mu_i)}} : i \text{ such that } u_i^* \neq 0 \right\}$$

and consider $\mathbf{f} - \alpha \mathbf{u}^*$. If $u_i^* = 0$, then

$$\|f_i - \alpha u_i^*\|_{L^1(\mu_i)} = \|f_i\|_{L^1(\mu_i)} = \lambda_i^{*\,1/(p-1)}.$$

If $u_i^* \neq 0$, then

$$|f_i(x) - \alpha u_i^*(x)| = \left| \frac{\lambda_i^{*1/(p-1)}}{\|u_i^*\|_{L^1(\mu_i)}} h_i^*(x)\,|u_i^*(x)| - \alpha u_i^*(x) \right|$$
$$= \frac{\lambda_i^{*1/(p-1)}}{\|u_i^*\|_{L^1(\mu_i)}} |u_i^*(x)| - \alpha h_i^*(x)\, u_i^*(x).$$

From (4) it now easily follows that

$$\|f_i - \alpha u_i^*\|_{L^1(\mu_i)} = \|f_i\|_{L^1(\mu_i)} = \lambda_i^{*1/(p-1)}.$$

Thus

$$\|\mathbf{f} - \alpha \mathbf{u}^*\|_{Y_p} = \|\mathbf{f}\|_{Y_p}$$

and $\alpha \mathbf{u}^*$ is also a best approximant to $\mathbf{f}$ from U. U is not a unicity space.

($\Leftarrow$) Assume U is not a unicity space. There therefore exists an $\mathbf{f} \in CY_p$ and $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ such that $\pm \mathbf{u}^*$ are best approximants to $\mathbf{f}$ from U. Now

$$2\,|f_i(x)| \leqslant |(f_i + u_i^*)(x)| + |(f_i - u_i^*)(x)|$$

for all $x \in K$, and $i = 1, ..., m$, which implies that

$$2\,\|f_i\|_{L^1(\mu_i)} \leqslant \|f_i + u_i^*\|_{L^1(\mu_i)} + \|f_i - u_i^*\|_{L^1(\mu_i)}$$

for $i = 1, ..., m$, and

$$2\,\|\mathbf{f}\|_{Y_p} \leqslant \|\mathbf{f} + \mathbf{u}^*\|_{Y_p} + \|\mathbf{f} - \mathbf{u}^*\|_{Y_p}.$$

Since

$$\|\mathbf{f}\|_{Y_p} = \|\mathbf{f} + \mathbf{u}^*\|_{Y_p} = \|\mathbf{f} - \mathbf{u}^*\|_{Y_p}$$

we must have equality throughout. Thus

$$2\,|f_i(x)| = |(f_i + u_i^*)(x)| + |(f_i - u_i^*)(x)| \tag{6.4}$$

for all $x \in K$ and $i = 1, ..., m$, and

$$\|f_i\|_{L^1(\mu_i)} = \|f_i + u_i^*\|_{L^1(\mu_i)} = \|f_i - u_i^*\|_{L^1(\mu_i)} \tag{6.5}$$

for $i = 1, ..., m$. From (6.4) we have $|f_i(x)| \geqslant |u_i^*(x)|$ for all $x \in K$ and $i = 1, ..., m$. Therefore $Z(f_i) \subseteq Z(u_i^*)$, $i = 1, ..., m$.

Since $\mathbf{0}$ is a best approximant to $\mathbf{f}$ from U, there exist $h_i^* \in L^\infty(K, \mu_i)$, $i = 1, ..., m$, satisfying

$$|h_i^*(x)| = 1, \qquad x \in K,\ i = 1, ..., m,$$

i.e., (1), with $h_i^*(x) = \operatorname{sgn} f_i(x)$ on $N(f_i)$ and

$$0 = \sum_{i=1}^{m} \|f_i\|_{L^1(\mu_i)}^{p-1} \int_K h_i^*(x)\, u_i(x)\, d\mu_i(x)$$

for all $\mathbf{u} \in U$. Setting $\lambda_i^* = \|f_i\|_{L^1(\mu_i)}^{p-1}$, $i = 1, ..., m$, we get that (3) holds. Note that $\lambda_i^* \geqslant 0$, and if $\lambda_i^* = 0$, then $f_i = 0$ and since $Z(f_i) \subseteq Z(u_i^*)$, we have $u_i^* = 0$. Thus $\lambda_i^* > 0$ if $\|u_i^*\|_{L^1(\mu_i)} > 0$.

Again, since $Z(f_i) \subseteq Z(u_i^*)$, we have that $h_i^*(x)\,|u_i^*(x)| \in C(K)$, $i = 1, ..., m$, i.e., (2) holds.

Finally, from (6.5),

$$\begin{aligned}
\|f_i\|_{L^1(\mu_i)} &= \|f_i \pm u_i^*\|_{L^1(\mu_i)} \\
&= \int_K |(f_i \pm u_i^*)(x)|\, d\mu_i(x) \\
&= \int_K h_i^*(x)(f_i \pm u_i^*)(x)\, d\mu_i(x) \\
&= \|f_i\|_{L^1(\mu_i)} \pm \int_K h_i^*(x)\, u_i^*(x)\, d\mu_i(x).
\end{aligned}$$

Thus

$$\int_K h_i^*(x)\, u_i^*(x)\, d\mu_i(x) = 0, \qquad i = 1, ..., m,$$

and (4) holds. ∎

Let us consider some examples.

EXAMPLE 1. $\dim U = 1$. Let $U = \operatorname{span}\{\mathbf{u}^*\}$, $\mathbf{u}^* = (u_1^*, ..., u_m^*)$. From Theorem 6.3, U is *not* a unicity space if and only if there exist $h_i^* \in L^\infty(K, \mu_i)$, $i = 1, ..., m$, satisfying

(1′) $|h_i^*(x)| = 1$, $x \in K$, $i = 1, ..., m$

(2′) $h_i^*(x)\,|u_i^*(x)| \in C(K)$, $i = 1, ..., m$

(4′) $\int_K h_i^*(x)\, u_i^*(x)\, d\mu_i(x) = 0$, $i = 1, ..., m$.

For each $i \in \{1, ..., m\}$, (1′), (2′), and (4′) are totally equivalent to the fact that if $u_i^* \neq 0$, then $\operatorname{span}\{u_i^*\}$ is not a unicity space for continuous functions with norm $L^1(K, \mu_i)$ (see Theorem 3.6). Thus U is a unicity space for CY_p if and only if there exists an $i \in \{1, ..., m\}$ such that $u_i^* \neq 0$, and $\operatorname{span}\{u_i^*\}$ is a unicity space for continuous functions with norm $L^1(K, \mu_i)$.

EXAMPLE 2. Simultaneous Approximation. Assume $\mu_i = \mu$, $i = 1, ..., m$, and for each $\mathbf{u} \in U$, $u_i = u$, $i = 1, ..., m$. Set

$$\tilde{U} = \{u : \mathbf{u} = (u, ..., u) \in U\}.$$

From Theorem 6.3, U is *not* a unicity space if and only if there exist $h_i \in L^\infty(K, \mu)$, $i = 1, ..., m$, a $u^* \in \tilde{U} \setminus \{0\}$, and $\lambda_i^* > 0$, $i = 1, ..., m$, satisfying

(1″) $|h_i^*(x)| = 1$, $x \in K$, $i = 1, ..., m$

(2″) $h_i^*(x) \, |u^*(x)| \in C(K)$, $i = 1, ..., m$

(3″) $\int_K (\sum_{i=1}^m \lambda_i^* h_i^*(x)) \, u(x) \, d\mu(x) = 0$ all $u \in \tilde{U}$

(4″) $\int_K h_i^*(x) \, u^*(x) \, d\mu(x) = 0$, $i = 1, ..., m$.

In the previous example we considered the case $n = 1$. Let us now assume that $n \geqslant 2$ and $m \geqslant 2$. Then *no* U is a unicity space in the problem of simultaneous approximation in CY_p.

A proof goes as follows. Since $n \geqslant 2$, there exists a $u^* \in \tilde{U}$ such that

$$\int_K u^*(x) \, d\mu(x).$$

Set $h_i^*(x) = 1$, $i = 1, ..., m-1$, and $h_m^*(x) = -1$, all $x \in K$. Thus (1″), (2″), and (4″) hold. The $\lambda_i^* > 0$ are simply chosen so that (3″) holds.

EXAMPLE 3. Tensor Product. If

$$U = U^1 \oplus \cdots \oplus U^m,$$

where $\mathbf{u} \in U^i$ implies $u_j = 0$ for all $j \neq i$, then

$$\min_{\mathbf{u} \in U} \|\mathbf{f} - \mathbf{u}\|_{Y_p} = \left(\sum_{i=1}^m \left[\min_{u_i \in U^i} \|f_i - u_i\|_{L^1(\mu_i)} \right]^p \right)^{1/p}.$$

Thus U is a unicity space for CY_p if and only if each U^i is a unicity space for continuous functions with norm $L^1(K, \mu_i)$, $i = 1, ..., m$.

One major question which remains unanswered is that of characterizing the U which are unicity spaces for CY_p for all $\mu_1, ..., \mu_m$ in $\mathscr{A}$.

For $n = 1$, i.e., $U = \text{span}\{\mathbf{u}^*\}$, it follows by the above that U has this property if and only if for some $i \in \{1, ..., m\}$, $u_i^* \neq 0$ and $\text{span}\{u_i^*\}$ is a unicity space for continuous functions with norm $L^1(K, \mu_i)$ for all μ_i in $\mathscr{A}$. This in turn (see Pinkus [28]) is equivalent to the fact that the support of u_i^* is a connected set. In the case of tensor products, as in Example 3, U is a unicity space for CY_p for all $\mu_1, ..., \mu_m$ in $\mathscr{A}$ if and only if each of the U^i have this same property.

We do not know the answer in general. Certain different phenomena seem to occur, the most obvious of which is exemplified by this next example. Consider

$$U = \operatorname{span}\{(1, x), (x, 1)\}$$

on $K = [-1, 1]$. U is a unicity space for CY_p for all measures in $\mathscr{A}$. This follows from the fact that for each $\mathbf{u} \in U \setminus \{\mathbf{0}\}$, the support of either u_1 or u_2 is a connected interval. As such there do not exist h_1, h_2 satsfying (1), (2), and (4) for both u_1 and u_2.

7. $A(\infty, q)$, $1 < q < \infty$

Let Y_i be a normed linear space with norm $\|\cdot\|_{Y_i}$, $i = 1, ..., m$. By Y we denote the normed linear space

$$Y = \{\mathbf{f} = (f_1, ..., f_m) : f_i \in Y_i, i = 1, ..., m\}$$

with norm

$$\|\mathbf{f}\|_Y = \max_{i=1,...,m} \|f_i\|_{Y_i}.$$

If Y^* is the dual space to Y, then

$$Y^* = \{\mathbf{h} = (h_1, ..., h_m) : h_i \in Y_i^*, i = 1, ..., m\}$$

with norm

$$\|\mathbf{h}\|_{Y^*} = \sum_{i=1}^{m} \|h_i\|_{Y_i^*}.$$

We first consider characterizations of best approximants. This is most easily stated under the assumption that each Y_i is *smooth*. For now we therefore make this assumption. For each $f \in Y_i$, $f \neq 0$, we let $h_f \in Y_i^*$ denote the unique linear functional satisfying $\|h_f\|_{Y_i^*} = 1$, and $h_f(f) = 1$. From

$$\begin{aligned} \mathbf{h}(\mathbf{f}) &= \sum_{i=1}^{m} h_i(f_i) \leqslant \sum_{i=1}^{m} \|f_i\|_{Y_i} \|h_i\|_{Y_i^*} \\ &\leqslant \left(\max_{i=1,...,m} \|f_i\|_{Y_i}\right)\left(\sum_{i=1}^{m} \|h_i\|_{Y_i^*}\right) = \|\mathbf{f}\|_Y \|\mathbf{h}\|_{Y^*}, \end{aligned}$$

we have that if $\mathbf{f} \in Y$, $\mathbf{f} \neq \mathbf{0}$, and $\mathbf{h} \in Y^*$ satisfies

$$\|\mathbf{h}\|_{Y^*} = 1, \qquad \mathbf{h}(\mathbf{f}) = \|\mathbf{f}\|_Y$$

then

(a) $h_i = d_i h_{f_i}$ with $d_i \geqslant 0$.

(b) $d_i = 0$ unless $i \notin J = \{j : \|f_j\|_{Y_j} = \|\mathbf{f}\|_Y\}$

(c) $\sum_{i \in J} |d_i| = 1$.

A characterization theorem is as follows.

THEOREM 7.1. *Let U be an n-dimensional subspace of Y. For $\mathbf{f} \in Y$, we have that $\mathbf{u}^* \in U$ is a best approximant to $\mathbf{f}$ from U if and only if there exist $j_1, ..., j_k \in J$, where*

$$J = \{l : \|f_l - u_l^*\|_{Y_l} = \|\mathbf{f} - \mathbf{u}^*\|_Y\},$$

and positive numbers $\lambda_1, ..., \lambda_k$ with $1 \leqslant k \leqslant \min\{n+1, m\}$, such that

$$\sum_{i=1}^{k} \lambda_i h_{f_{j_i} - u_{j_i}^*}(u_{j_i}) = 0$$

for all $\mathbf{u} \in U$.

In the consideration of the uniqueness property, the smoothness plays no role. We therefore drop this assumption.

Set

$$P_i U = \{u_i : \mathbf{u} = (u_1, ..., u_m) \in U\}.$$

Thus $0 \leqslant \dim P_i U \leqslant n$ for each $i = 1, ..., m$. Necessary and sufficient conditions for the unicity of U are easily stated and proved if the norms on the Y_i are strictly convex for each i. We divide the result into two parts in order to emphasize that in one direction this condition is not necessary (and so that we need not repeat it in Section 11). In what follows we assume that each Y_i is a space of dimension $\geqslant n$.

PROPOSITION 7.2. *The n-dimensional subspace U of Y is not a unicity space for Y if* $\dim P_i U < n$ *for some $i = 1, ..., m$.*

Proof. Assume $\dim P_i U < n$. Then there exists a $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ such that $u_i^* = 0$.

Let $f_i \in Y_i$, $f_i \neq 0$, be such that 0 is a best approximant to f_i from $P_i U$. Such an f_i exists since $\dim Y_i \geqslant n$. Assume, without loss of generality, that $\|f_i\|_{Y_i} = 1$. Set

$$\mathbf{f} = (0, ..., 0, f_i, 0, ..., 0).$$

Then $\mathbf{0}$ is a best approximant to $\mathbf{f}$ from Y since

$$1 = \|\mathbf{f}\|_Y = \|f_i\|_{Y_i} \leqslant \|f_i - u_i\|_{Y_i} \leqslant \|\mathbf{f} - \mathbf{u}\|_Y$$

for every $\mathbf{u} \in U$. Furthermore, since $u_i^* = 0$,

$$\|\mathbf{f} - \alpha \mathbf{u}^*\|_Y = \|\mathbf{f}\|_Y$$

for every α such that

$$|\alpha| \, \|u_j^*\|_{Y_j} \leqslant 1, \qquad j = 1, ..., m.$$

Thus U is not a unicity space. ∎

THEOREM 7.3. *Assume that each Y_i is a strictly convex normed linear space, $i = 1, ..., m$. Then U is a unicity space if and only if* $\dim P_i U = n$ $(=\dim U)$ *for each $i = 1, ..., m$.*

Proof. From the previous proposition it remains to prove that if U is not a unicity space, then $\dim P_l U < n$ for some l.

Assume $\mathbf{f} \in Y$, $\|\mathbf{f}\|_Y = 1$, and $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ is such that $\pm \mathbf{u}^*$ are best approximants to $\mathbf{f}$ from Y.

Thus

$$1 = \|\mathbf{f}\|_Y = \|\mathbf{f} \pm \mathbf{u}^*\|_Y.$$

Let

$$l \in J = \{ j : \|f_j\|_{Y_j} = \|\mathbf{f}\|_Y \}.$$

Then

$$\begin{aligned} 1 = \|f_l\|_{Y_l} &\leqslant \tfrac{1}{2} \|f_l + u_l^*\|_{Y_l} + \tfrac{1}{2} \|f_l - u_l^*\|_{Y_l} \\ &\leqslant \tfrac{1}{2} \|\mathbf{f} + \mathbf{u}^*\|_Y + \tfrac{1}{2} \|\mathbf{f} - \mathbf{u}^*\|_Y \\ &= 1. \end{aligned}$$

Thus

$$1 = \|f_l\|_{Y_l} = \|f_l - u_l^*\|_{Y_l} = \|f_l + u_l^*\|_{Y_l}.$$

Since the norm Y_l is strictly convex, it follows that

$$f_l - u_l^* = f_l + u_l^*.$$

Thus $u_l^* = 0$. But $\mathbf{u}^* \neq \mathbf{0}$. Therefore $\dim P_l U < n$. ∎

Remark. Theorem 7.3 should be considered together with Theorem 16.2. It is really a special case (although not explicitly stated) of a result of Zuhovitsky and Stechkin [36].

Remark. Note that if $Y_i = Y$, and Y is a strictly convex normed linear space, then in the problem of simultaneous approximation U is always a unicity space. On the other hand, in the problem of tensor product approximation, U is never a unicity space.

8. $A(p, \infty)$, $1 < p < \infty$

When dealing with the L^∞-norm we always restrict ourselves to the space of continuous functions. As such we assume, as in Section 3, that D is a compact Hausdorff set and $C(D)$ the space of continuous real-valued functions defined on D with the usual uniform norm. We let $Y = A(p, \infty)$ denote the normed linear space

$$Y = \{\mathbf{f} = (f_1, ..., f_m) : f_i \in C(D)\}$$

with norm

$$\|\mathbf{f}\|_Y = \left(\sum_{i=1}^{m} \|f_i\|_\infty^p \right)^{1/p}.$$

Thus the dual space Y^* is given by

$$Y^* = \{\mu = (\mu_1, ..., \mu_m) : \mu_i \in C^*(D),\ i = 1, ..., m\}$$

with norm

$$\|\mu\|_{Y^*} = \left(\sum_{i=1}^{m} \|\mu_i\|_{T.V.}^{p'} \right)^{1/p'},$$

where $1/p + 1/p' = 1$, and $\|\mu_i\|_{T.V.}$ denotes the total variation of the measure μ_i.

Applying results of Sections 2, 3, and 4 (Theorems 2.2 and 3.2, and Proposition 4.1), we obtain the following characterization of best approximants.

THEOREM 8.1. *Let U be an n-dimensional subspace of Y. Then $\mathbf{u}^*$ is a best approximant to $\mathbf{f}$ from U if and only if for some k, $1 \leqslant k \leqslant n + 1$,* *there exist points* $\{x_i^j\}_{i=1}^{m}{}_{j=1}^{k}$ *in* D, *positive numbers* λ_j, $j=1, ..., k$, *and* $\varepsilon_{ij} \in \{-1, 1\}$, $j=1, ..., k$, $i=1, ..., m$, *such that*

(1) $\sum_{i=1}^{m} \sum_{j=1}^{k} \lambda_j \|f_i - u_i^*\|_\infty^{p-1} \varepsilon_{ij} u_i(x_i^j) = 0$ *all* $\mathbf{u} \in U$

(2) $\varepsilon_{ij}((f_i - u_i^*)(x_i^j)) = \|f_i - u_i^*\|_\infty$, $j=1, ..., k$, $i=1, ..., m$.

The problem of characterizing unicity spaces in this case seems to be a difficult one. One partial result is the following.

PROPOSITION 8.2. *If* U *is not a unicity space for* Y, *then there exists a* $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ *and* $\{x_i\}_{i=1}^{m} \in D$, *such that*

$$u_i^*(x_i) = 0, \qquad i=1, ..., m. \tag{8.1}$$

Proof. Since U is not a unicity space, there exists an $\mathbf{f} \in Y$ and $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ such that both $\pm\mathbf{u}^*$ are best approximants to $\mathbf{f}$ from U. For each $i=1, ..., m$,

$$2\|f_i\|_\infty \leqslant \|f_i - u_i^*\|_\infty + \|f_i + u_i^*\|_\infty. \tag{8.2}$$

Thus

$$2\|\mathbf{f}\|_Y \leqslant \|\mathbf{f} - \mathbf{u}^*\|_Y + \|\mathbf{f} + \mathbf{u}^*\|_Y. \tag{8.3}$$

Since $\pm\mathbf{u}^*$ are best approximants, we must have equality in (8.3). The strict convexity of the l_p^m-norm ($1 < p < \infty$) implies that we have equality in (8.2) for all i, and

$$\|f_i - u_i^*\|_\infty = c\,\|f_i + u_i^*\|_\infty, \qquad i=1, ..., m,$$

for some $c \geqslant 0$. Since $\|\mathbf{f} - \mathbf{u}^*\|_Y = \|\mathbf{f} + \mathbf{u}^*\|_Y$ we have $c=1$. Thus

$$\|f_i\|_\infty = \|f_i - u_i^*\|_\infty = \|f_i + u_i^*\|_\infty \tag{8.4}$$

for each i.

If $x_i \in D$ is such that $|f_i(x_i)| = \|f_i\|_\infty$, then from (8.4) we must have $u_i^*(x_i) = 0$. ∎

If (8.1) does not hold for any $\mathbf{u} \in U \setminus \{\mathbf{0}\}$, then U is necessarily a unicity space for Y. The converse does not in general hold. A partial result in the converse direction is the following much more demanding condition. (Note that the strict convexity of l_p^m is not used here.) We recall that for $u \in C(D)$, $Z(u)$ is its zero set. By $|Z(u)|$ we denote the number of zeros of u in D.

PROPOSITION 8.3. *If there exists a* $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ *such that*

$$|Z(u_i^*)| \geq \dim P_i U$$

for each $i = 1, ..., m$, *then* U *is not a unicity space for* Y.

Proof. If $u_i^* = 0$, then let f_i be any function in $C(D)$ for which the zero function is a best uniform approximant to f_i from $P_i U$.

Assume $u_i^* \neq 0$. By assumption $P_i U$ is not a Haar space. However, more is true. By a standard construction, due to Achiezer [1, p. 68] in his proof of the Haar Theorem, there exists an $f_i \in C(D)$ such that αu_i^* is a best approximant to f_i from $P_i U$ for each $|\alpha| \leq 1$.

It now follows that $\alpha \mathbf{u}^*$ is a best approximant to $\mathbf{f} = (f_1, ..., f_m)$ from U for each $|\alpha| \leq 1$. Thus U is not a unicity space for Y. ∎

EXAMPLE 1. $\dim U = 1$. By putting together Propositions 8.2 and 8.3, we get that $U = \text{span}\{\mathbf{u}^*\}$ is a unicity space for Y if and only if there do not exist points $\{x_i\}_{i=1}^m$ in D such that $u_i^*(x_i) = 0$, $i = 1, ..., m$.

Remark. It is interesting to juxtapose this result with the analogous results for $p = 1$ and $p = \infty$. For $p = 1$, we prove in Proposition 10.3 that $U = \text{span}\{\mathbf{u}^*\}$ is a unicity space if and only if there do not exist $\{x_i\}_{i=1}^m$ in D and $\varepsilon_i \in \{-1, 1\}$ such that $\sum_{i=1}^m \varepsilon_i u_i^*(x_i) = 0$. For $p = \infty$, see Section 12, we have that U is a unicity space if and only if u_i^* has no zero in D for each $i = 1, ..., m$. That is, there does not exist an $i \in \{1, ..., m\}$ and $x_i \in D$ such that $u_i^*(x_i) = 0$.

EXAMPLE 2. Tensor Product. If $U = U^1 \oplus \cdots \oplus U^m$, then it follows by definition that U is a unicity space for Y if and only if each U^i is a Haar space (unicity space for $C(D)$).

EXAMPLE 3. Simultaneous Approximation. Let $\tilde{U}$ be an n-dimensional subspace of $C(D)$ and

$$U = \{\mathbf{u} : \mathbf{u} = (u, ..., u), u \in \tilde{U}\}.$$

We have the following result. (Note the demand that $n > 1$.)

PROPOSITION 8.4. *Let* $n > 1$ *(and* $m > 1$*). Then for* U *and* $\tilde{U}$ *as above,* U *is not a unicity space for* Y.

Proof. Since $n \geq 2$, there exists a $u^* \in \tilde{U} \setminus \{0\}$ which vanishes at some $x^* \in D$. There exists an $f \in C(D)$ satisfying $f(x^*) = \|f\|_\infty = 1$, and such that αu^* is a best approximant to f from $\text{span}\{u^*\}$ for all $|\alpha| \leq 1$ (normalize u^*

if necessary). Choose any numbers $\{a_i\}_{i=1}^m$ satisfying $|a_i| \geqslant 1$, $i=1, ..., m$, and $\sum_{i=1}^m |a_i|^{p-1} \operatorname{sgn} a_i = 0$ $(m \geqslant 2)$. Thus

$$\sum_{i=1}^m \|a_i f\|_\infty^{p-1} (\operatorname{sgn} a_i)\, u(x^*) = u(x^*) \sum_{i=1}^m |a_i|^{p-1} \operatorname{sgn} a_i = 0 \tag{8.5}$$

for all $u \in \tilde{U}$. Furthermore

$$(\operatorname{sgn} a_i)(a_i f)(x^*) = \|a_i f\|_\infty, \qquad i=1, ..., m. \tag{8.6}$$

From (8.5), (8.6), and Theorem 8.1, it follows that $\mathbf{0}$ is a best approximant to

$$\mathbf{f}_\mathbf{a} = (a_1 f, ..., a_m f)$$

from U. (Here $k=1$, $x_i^j = x^*$, $\varepsilon_{ij} = \operatorname{sgn} a_i$.) Now for each $i \in \{1, ..., m\}$

$$\|a_i f - u^*\|_\infty = |a_i|\; \|f - (u^*/a_i)\|_\infty = |a_i| = \|a_i f\|_\infty$$

since $1/|a_i| \leqslant 1$. Thus U is not a unicity space. ∎

Remark. From Proposition 8.4 we see that the converse of Proposition 8.3 is not valid.

9. $A(1, 1)$ $(B(1, 1))$

A much simpler case is given by

$$\|\mathbf{f}\|_{A(1,1)} = \sum_{i=1}^m \|f_i\|_1,$$

where $\mathbf{f} = (f_1, ..., f_m)$, and

$$\|f_i\|_1 = \int_{D_i} |f_i(x)|\, dv_i(x), \qquad i=1, ..., m,$$

where the D_i and v_i satisfy the conditions in Section 6. Contained in this case is the $B(1, 1)$-norm. This is a simpler case because we should consider this norm as a standard L^1-norm having nothing to do with vector-valued functions. It is as if we are given a function f which is f_i on D_i, for $i=1, ..., m$, and we take its L^1-norm. Since we are dealing with the usual L^1-norm, the theory of L^1-approximation as expounded upon in Pinkus [29] is applicable.

Assuming that each v_i is a non-atomic positive measure, $i=1, ..., m$, it follows from Theorem 3.5 that no finite dimensional subspace U is a unicity space. If, in addition, we have that $D_i = K \subset \mathbb{R}^d$ is compact, $K = \overline{\operatorname{int} K}$, and each μ_i is in $\mathscr{A}$, then if U is a finite dimensional subspace such that $u_i \in C(K)$ for each $i = 1, ..., m$, then conditions for U to be a unicity space with respect to the $\mathbf{f}$ satisfying $f_i \in C(K)$, $i = 1, ..., m$, are given in Theorems 3.6 and 3.7. Similarly U is a unicity space in the above problem for every choice of measures $\mu_1, ..., \mu_m$ in $\mathscr{A}$ if and only if U satisfies Property A, properly translated into this context. Using Theorem 3.9(2), we see that this means that U must be of the form

$$U = U^1 \oplus \cdots \oplus U^m,$$

i.e., a tensor product, and each U^i satisfies Property A.

There are some problems which remain interesting despite the fact that this is a special case of a well studied problem. It is natural, for example, to consider the case where $D_i = K$ and $\nu_i = \mu$ for all $i = 1, ..., m$. What are then the conditions on U (as in the above problems) so that it is a unicity space for every $\mu \in \mathscr{A}$? We have no answer at present, except in the interesting case of Simultaneous Approximation. That is, where in addition $u_i = u$ for each $i = 1, ..., m$, and $\mathbf{u} \in U$. Here the conditions of Theorem 3.6 may be stated as follows.

There do *not* exist $h_i \in L^\infty(K, \mu)$, $i = 1, ..., m$, and $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ for which

(1) $|h_i(x)| = 1$ for all $x \in K$, $i = 1, ..., m$

(2) $h_i |u^*| \in C(K)$, $i = 1, ..., m$

(3) $\int_K (\sum_{i=1}^m h_i) u \, d\mu = 0$ for all $\mathbf{u} \in U$.

If m is even, there obviously do exist such h_i and $\mathbf{u}^*$. Just take $h_i = (-1)^i$, $i = 1, ..., m$, and let $\mathbf{u}^*$ be arbitrarily chosen. Thus in the problem of Simultaneous Approximation, U is never a unicity space for any $\mu \in \mathscr{A}$ if m even. For m odd, the situation is not as simple. If

$$\tilde{U} = \{u : \mathbf{u} = (u, ..., u) \in U\}$$

is not a unicity space, then neither is U. The converse does not seem to be necessarily true. But U is a unicity space for *all* $\mu \in \mathscr{A}$ in the above problem if and only if $\tilde{U}$ is a unicity space for *all* $\mu \in \mathscr{A}$. We state this formally.

PROPOSITION 9.1. *Let U and $\tilde{U}$ be as above, of odd dimension. Then U is a unicity space in the above problem for all $\mu \in \mathscr{A}$ if and only if $\tilde{U}$ is a unicity space for $C_1(K, \mu)$ for all $\mu \in \mathscr{A}$, i.e., satisfies Property* A.

Proof. We need only prove that if $\tilde{U}$ is a unicity space for all $\mu \in \mathscr{A}$, then so is U. $\tilde{U}$ satisfies Property A. That is, given any $u \in \tilde{U} \setminus \{0\}$ and $[K \setminus Z(u)] = \bigcup_{j=1}^r A_j$, and given any $\varepsilon_j \in \{-1, 1\}$, $j = 1, ..., r$, there exists a $v \in \tilde{U} \setminus \{0\}$ satisfying

(a) $v=0$ a.e. on $Z(u)$

(b) $\varepsilon_j v \geqslant 0$ on A_j for all j.

Now assume that U is not a unicity space for some $\mu^* \in \mathscr{A}$. Then there exist $h_i \in L^\infty(K, \mu^*)$, $i=1, ..., m$, and $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ such that

(1) $h_i(x)| = 1$ for all $x \in K$, $i=1, ..., m$

(2) $h_i |u^*| \in C(K)$

(3) $\int_K (\sum_{i=1}^m h_i)\, u \, d\mu^* = 0$ for all $\mathbf{u} \in U$.

From (1), (2), and since m is odd, it follows that on each component A_j of $K \setminus Z(u^*)$, the function $\sum_{i=1}^m h_i$ is a non-zero constant. Let ε_j denote its sign. By assumption there exists a $v \in \tilde{U} \setminus \{0\}$ such that

(a) $v=0$ a.e. on $Z(u^*)$

(b) $\varepsilon_j v \geqslant 0$ on A_j, all j.

Thus

$$\int_K \left(\sum_{i=1}^m h_i \right) v \, d\mu^* > 0.$$

This contradiction to (3) proves the proposition. ∎

10. $A(1, \infty)$

We assume that D is a compact Hausdorff set and $C(D)$ the space of continuous real-valued functions defined on D with the usual uniform norm. We let $Y = A(1, \infty)$ denote the normed linear space

$$Y = \{\mathbf{f} = (f_1, ..., f_m) : f_i \in C(D)\}$$

with norm

$$\|\mathbf{f}\|_Y = \sum_{i=1}^m \|f_i\|_\infty .$$

Thus the dual space Y^* is given by

$$Y^* = \{\mu = (\mu_1, ..., \mu_m) : \mu_i \in C^*(D), i = 1, ..., m\}$$

with norm

$$\|\mu\|_{Y^*} = \max_{i=1,...,m} \|\mu_i\|_{T.V.}.$$

Similiar arguments to those used in obtaining Theorem 8.1 lead to this next theorem.

THEOREM 10.1. *Let U be an n-dimensional subspace of Y. Then* $\mathbf{u}^*$ *is a best approximant to* $\mathbf{f}$ *from U if and only if for some k, $1 \leqslant k \leqslant n+1$, there exist points $\{x_i^j\}_{i=1}^{m}{}_{j=1}^{k}$ in D, positive numbers λ_j, $j=1, ..., k$, and $\varepsilon_{ij} \in \{-1, 1\}$, $j=1, ..., k$, $i=1, ..., m$, such that*

(1) $\sum_{i=1}^{m} \sum_{j=1}^{k} \lambda_j \varepsilon_{ij} u_i(x_i^j) = 0$ *all* $\mathbf{u} \in U$

(2) $\varepsilon_{ij}((f_i - u_i^*)(x_i^j)) = \|f_i - u_i^*\|_\infty$, $j=1, ..., k; i=1, ..., m$.

The uniqueness problem is not a simple one, and the results we have are partial. Proposition 8.3 holds in this setting since the proof thereof did not use the strict convexity of the norm. One other partial result is a consequence of this characterization.

PROPOSITION 10.2. *If U is not a unicity space for Y, then there exist $\{x_i\}_{i=1}^{m} \in D$, $\varepsilon_i \in \{-1, 1\}$, $i=1, ..., m$, and a* $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ *such that*

$$\sum_{i=1}^{m} \varepsilon_i u_i^*(x_i) = 0.$$

Proof. Assume U is not a unicity space. Let $\mathbf{f} \in Y$ and assume $\mathbf{0}$ and $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ are best approximants to $\mathbf{f}$ from U. Let $\{x_i^j\}_{i=1}^{m}{}_{j=1}^{k}$, $\{\lambda_j\}_{j=1}^{k}$, and $\{\varepsilon_{ij}\}_{i=1}^{m}{}_{j=1}^{k}$ be as in Therem 10.1. Thus

$$\varepsilon_{ij} f_i(x_i^j) = \|f_i\|_\infty, \qquad j=1, ..., k; i=1, ..., m,$$

and it also easily follows that

$$\varepsilon_{ij}((f_i - u_i^*)(x_i^j)) = \|f_i - u_i^*\|_\infty, \qquad j=1, ..., k; i=1, ..., m.$$

Thus

$$\varepsilon_{ij} u_i^*(x_i^j) = \|f_i\|_\infty - \|f_i - u_i^*\|_\infty$$

for each $j=1, ..., k$ and $i=1, ..., m$. Since

$$\sum_{i=1}^{m} \|f_i\|_\infty = \|\mathbf{f}\|_Y = \|\mathbf{f} - \mathbf{u}^*\|_Y = \sum_{i=1}^{m} \|f_i - u_i^*\|_\infty,$$

we have

$$\sum_{i=1}^{m} \varepsilon_{ij} u_i^*(x_i^j) = 0. \quad ▮$$

If for each $\mathbf{u} \in U \setminus \{\mathbf{0}\}$ we cannot find $x_i \in D$, $i = 1, ..., m$, and $\varepsilon_i \in \{-1, 1\}$ such that

$$\sum_{i=1}^{m} \varepsilon_i u_i(x_i) = 0,$$

then U is a unicity space for Y. In general the converse is not valid. However, the converse is valid if $\dim U = 1$.

EXAMPLE 1. $\dim U = 1$.

PROPOSITION 10.3. *If $U = \operatorname{span}\{\mathbf{u}^*\}$, i.e., $\dim U = 1$, then U is a unicity space if and only if there do not exist points $\{x_i\}_{i=1}^m$ in D, and $\varepsilon_i \in \{-1, 1\}$, $i = 1, ..., m$, such that*

$$\sum_{i=1}^{m} \varepsilon_i u_i^*(x_i) = 0. \tag{10.1}$$

Proof. One direction is contained in Proposition 10.2. Assume x_i and ε_i, as above, exist satisfying (10.1). Choose a constant $c > 0$ such that $c > 3\, \|u_i\|_\infty$, $i = 1, ..., m$. Set

$$f_i(x) = \varepsilon_i \left[c - |u_i^*(x) - u_i^*(x_i)|\right].$$

It follows by inspection that

$$\|f_i\|_\infty = c = \varepsilon_i f_i(x_i). \tag{10.2}$$

Using Theorem 10.1, Eqs. (10.1) and (10.2) imply that $\mathbf{0}$ is a best approximant to $\mathbf{f}$ from U.

Since $c > 3\, \|u_i^*\|_\infty$, we also have $\varepsilon_i((f_i - u_i^*)(x)) \geqslant 0$ for all $x \in D$. Now, for any $x \in D$,

$$\begin{aligned} |(f_i - u_i^*)(x)| &= \varepsilon_i((f_i - u_i^*)(x)) \\ &= c - |u_i^*(x) - u_i^*(x_i)| - \varepsilon_i u_i^*(x) \\ &\leqslant c - \varepsilon_i u_i^*(x_i) \\ &= \varepsilon_i((f_i - u_i^*)(x_i)) \\ &= |(f_i - u_i^*)(x_i)|. \end{aligned}$$

Thus

$$\|f_i - u_i^*\|_\infty = \varepsilon_i((f_i - u_i^*)(x_i)). \tag{10.3}$$

It now follows from (10.1) that $\mathbf{u}^*$ is also a best approximant to $\mathbf{f}$ from U. ∎

EXAMPLE 2. Tensor Product. If $U = U^1 \oplus \cdots \oplus U^m$, then it follows by definition that U is a unicity space for Y if and only if each U^i is a Haar space (unicity space for $C(D)$).

EXAMPLE 3. Simultaneous Approximation. Let $\tilde{U}$ be an n-dimensional subspace of $C(D)$, and

$$U = \{\mathbf{u} = (u, \ldots, u) : u \in \tilde{U}\}.$$

We have the following simple result.

PROPOSITION 10.4. *For U and $\tilde{U}$ as above, U is not a unicity space for Y if m is even or U is not a Haar space.*

Proof. Assume m is even. If $n = 1$, we can apply Proposition 10.3. Let $x_i = x$ be any point in D, and $\varepsilon_i = (-1)^i$, $i = 1, \ldots, m$. Then $\sum_{i=1}^m \varepsilon_i = 0$ and (10.1) holds.

Assume $n \geqslant 2$. Then there exists a $u^* \in \tilde{U} \setminus \{0\}$ which vanishes at some point of D. Thus $\operatorname{span}\{u^*\}$ is not a Haar space in $C(D)$. Let $f \in C(D)$ be such that 0 and u^* are best approximants to f from $\operatorname{span}\{u^*\}$. Set

$$\mathbf{f} = (f, -f, f, \ldots, -f).$$

Since 0 and u^* are best approximants to f from $\operatorname{span}\{u^*\}$,

$$\|\mathbf{f} - \mathbf{u}^*\|_Y = \|\mathbf{f}\|_Y = m\, \|f\|_X.$$

For any $\mathbf{u} \in U$,

$$\|\mathbf{f} - \mathbf{u}\|_Y = \sum_{i=1}^m \|(-1)^{i+1} f - u\|_X.$$

Now

$$\|f - u\|_X + \|-f - u\|_X \geqslant 2\, \|f\|_X,$$

and since m is even, we obtain

$$\|\mathbf{f} - \mathbf{u}\|_Y \geqslant m\, \|f\|_X$$

for any $\mathbf{u} \in U$. Thus $\mathbf{0}$, $\mathbf{u}^*$ are best approximants to $\mathbf{f}$ from U.

If $\tilde{U}$ is not a Haar space, we apply Proposition 8.3. ∎

The converse of Proposition 10.4 is not valid, at least in the case $n=1$. That is, it may be that m is odd and $\tilde{U}$ is a Haar space, and yet U is, or is not a unicity space. This follows from Proposition 10.3. For $n=1$, a necessary and sufficient condition is given by the existence of points $\{x_i\}_{i=1}^m$ in D, and $\varepsilon_i \in \{-1, 1\}$ such that

$$\sum_{i=1}^{m} \varepsilon_i u^*(x_i) = 0, \tag{10.4}$$

where $U = \operatorname{span}\{\mathbf{u}^*\}$. Now $\tilde{U} = \operatorname{span}\{u^*\}$ is a Haar space of dimension 1 if and only if u^* does not vanish on its domain of definition. This condition (and m odd) is insufficient to determine whether (10.4) will or will not hold.

Proposition 10.4 should be contrasted with Proposition 8.4. It was rather surprising to us that we were unable to prove a result as strong as Proposition 8.4 in this weaker setting.

11. $A(\infty, 1)$

We let $Y = A(\infty, 1)$ denote the Banach space

$$Y = \{\mathbf{f} = (f_1, ..., f_m) : f_i \in L^1(D_i, v_i), i = 1, ..., m\}$$

with norm

$$\|\mathbf{f}\|_Y = \max_{i=1,...,m} \|f_i\|_1 = \max_{i=1,...,m} \int_{D_i} |f_i(x)| \, dv_i(x),$$

where the D_i and v_i satisfy the conditions of Section 6. The dual space Y^* may be identified with

$$Y^* = \{\mathbf{h} = (h_1, ..., h_m) : h_i \in L^{\infty}(D_i, v_i), i = 1, ..., m\}$$

with norm

$$\|\mathbf{h}\|_{Y^*} = \sum_{i=1}^{m} \|h_i\|_{\infty}.$$

For convenience we assume that $D = K \subset \mathbb{R}^d$ is compact, $K = \overline{\operatorname{int} K}$, and $v_i = \mu_i \in \mathscr{A}$.

THEOREM 11.1. *Let U be an n-dimensional subspace of Y. Then $\mathbf{u}^*$ is a best approximant to $\mathbf{f}$ from U if and only if there exist k, $1 \leqslant k \leqslant \min\{m, n+1\}$, distinct $j_1, ..., j_k \in J$, where*

$$J = \{l : \|f_l - u_l^*\|_1 = \|\mathbf{f} - \mathbf{u}^*\|_Y\},$$

positive numbers $\lambda_1, \ldots, \lambda_k$, *and* $h_{j_i} \in L^{\infty}(K, \mu_{j_i})$, *satisfying*

(1) $|h_{j_i}(x)| = 1$, *all* $x \in K$, $i = 1, \ldots, k$

(2) $\int_K h_{j_i}(x)(f_{j_i} - u^*_{j_i})(x)\, d\mu_{j_i} = \|f_{j_i} - u^*_{j_i}\|_1$, $i = 1, \ldots, k$

(3) $\sum_{i=1}^k \lambda_i \int_K h_{j_i}(x) u_{j_i}(x)\, d\mu_{j_i}(x) = 0$ *all* $\mathbf{u} \in U$.

In the consideration of the unicity property we restrict ourselves to continuous functions, see, e.g., Section 6, and let CY denote the restriction of Y to $\mathbf{f} = (f_1, \ldots, f_m)$ with $f_i \in C(K)$, $i = 1, \ldots, m$. Without this restriction we easily obtain an analogue of Proposition 6.2. Two simple necessary (but not sufficient) conditions for unicity are contained in this next proposition.

PROPOSITION 11.2. *Let U be an n-dimensional subspace of CY. Set*

$$P_i U = \{u_i : \mathbf{u} = (u_1, \ldots, u_m) \in U\}.$$

If, for some $i \in \{1, \ldots, m\}$, $P_i U$ is not a unicity space for $C_1(K, \mu_i)$ of dimension n, then U is not a unicity space for CY.

Proof. If $\dim P_i U < n$ for some $i \in \{1, \ldots, m\}$, then we appeal to Proposition 7.2. Assume $P_i U$ is not a unicity space for $C_1(K, \mu_i)$ for some $i \in \{1, \ldots, m\}$. Let $f_i \in C(K)$, normalized so that $\|f_i\|_1 = 1$, be such that $\pm u_i^* \in P_i U \setminus \{0\}$ are best approximants to f_i from $P_i U$ in the $L^1(K, \mu_i)$ norm. Let $\mathbf{f} = (f_1, \ldots, f_m)$ where $f_j = 0$ for $j \neq i$. Then for all $\mathbf{u} \in U$, $\|\mathbf{f} - \mathbf{u}\|_Y \geqslant \|f_i - u_i\|_1 \geqslant \|f_i\|_1 = 1$. For ε small, such that $|\varepsilon| \cdot \|u_j^*\|_1 \leqslant 1$, $j \in \{1, \ldots, m\} \setminus \{i\}$, and $|\varepsilon| \leqslant 1$, we obtain $\|\mathbf{f} - \varepsilon \mathbf{u}^*\|_Y = \|f_i - \varepsilon u_i^*\|_1 = 1$. Thus U is not a unicity space. ∎

EXAMPLE 1. $\dim U = 1$. Let $U = \operatorname{span}\{\mathbf{u}^*\}$. In this case the converse of Proposition 11.2 is valid. That is, if for each $i \in \{1, \ldots, m\}$, $u_i^* \neq 0$, and $\operatorname{span}\{u_i^*\}$ is a unicity space for $C_1(K, \mu_i)$, then U is a unicity space. To see this, assume U is not a unicity space. Let $\mathbf{f} \in CY$ be such that $\pm \mathbf{u}^*$ are best approximants of $\mathbf{f}$. Let $i \in \{1, \ldots, m\}$ be such that $\|f_i\|_1 = \|\mathbf{f}\|_Y$. Then it is easily checked that

$$\|f_i\|_1 = \|f_i - \alpha u_i^*\|_1$$

for all $|\alpha| \leqslant 1$. This in turn implies that

$$\|f_i\|_1 = \min_{\sigma} \|f_i - \sigma u_i^*\|_1.$$

Thus $\pm u_i^*$ are best approximants to f_i from $\operatorname{span}\{u_i^*\}$, which is a contradiction.

EXAMPLE 2. Tensor Product. From Proposition 11.2 ($m \geqslant 2$), U cannot be a unicity space.

EXAMPLE 3. Simultaneous Approximation. We assume that $\mu_i = \mu$, $i = 1, \dots, m$, $\tilde{U}$ is an n-dimensional subspace of $C(K)$, and

$$U = \{\mathbf{u} = (u, \dots, u) : u \in \tilde{U}\}.$$

PROPOSITION 11.3. *For U and $\tilde{U}$ as above, and $n, m \geqslant 2$, U is not a unicity space for CY.*

Proof. Let

$$\int_K 1 \, d\mu = c > 0.$$

Since $n \geqslant 2$, there exists a $u^* \in \tilde{U} \setminus \{0\}$ satisfying $\|u^*\|_1 \leqslant c$, $\|u^*\|_\infty \leqslant 1$, and

$$\int_K u^* \, d\mu = 0.$$

Set $\mathbf{f} = (f_1, \dots, f_m)$, where $f_1 = 1$, $f_2 = -1$, and $f_i = 0$, $i \geqslant 3$ ($m \geqslant 2$). Thus $\|\mathbf{f}\|_Y = c$. For $\mathbf{u} \in U$

$$\begin{aligned} \|\mathbf{f} - \mathbf{u}\|_Y &\geqslant \max\{\|f_1 - u\|_1, \|f_2 - u\|_1\} \\ &= \max\{\|f_1 - u\|_1, \|f_1 + u\|_1\} \geqslant \|f_1\|_1 = c. \end{aligned}$$

Let $\alpha \in [-1, 1]$. For $i \in \{1, 2\}$, since $\|u^*\|_\infty \leqslant 1$,

$$\|f_i - \alpha u^*\|_1 = \int_K (1 + (-1)^i \alpha u^*) \, d\mu = \int_K 1 \, d\mu = c.$$

For $i \geqslant 3$,

$$\|f_i - \alpha u^*\|_1 = \|\alpha u^*\|_1 \leqslant c.$$

Thus $\alpha \mathbf{u}^*$ are best approximants to $\mathbf{f}$ from U for all $\alpha \in [-1, 1]$. ∎

12. $A(\infty, \infty)$ ($B(\infty, \infty)$)

As in Section 8, D is a compact Hausdorff set and $C(D)$ the space of continuous real-valued functions defined on D with the usual uniform norm. We let $Y = A(\infty, \infty)$ denote the normed linear space

$$Y = \{\mathbf{f} = (f_1, \dots, f_m) : f_i \in C(D)\}$$

with norm

$$\|\mathbf{f}\|_Y = \max_{i=1,\dots,m} \|f_i\|_\infty = \max_{\substack{i=1,\dots,m\\ x\in D}} |f_i(x)|.$$

Contained in this case is the $B(\infty, \infty)$-norm. This is a simpler case because this can and should be reinterpreted as a space of real-valued rather than vector-valued functions. As such it follows that U is a unicity space if and only if it is a Haar space, in the above sense. This simply translates as follows: U is a unicity space for Y if and only if for each $\mathbf{u} \in U \setminus \{\mathbf{0}\}$, the sum over i in $\{1, \dots, m\}$ of the numbers of zeros of u_i is at most $\dim U - 1$. It therefore follows that if U is a tensor product space, it is not a unicity space, and that in the case of simultaneous approximation with $m \geqslant 2$ and $\dim U \geqslant 2$, U is not a unicity space.

PART B

13. The $B(p, q)$-Norm: General Results

The $B(p, q)$-norm is defined by

$$\|\mathbf{f}\|_{B(p,q)} = \left(\int_D \left(\sum_{i=1}^m |f_i(x)|^q \right)^{p/q} dv(x) \right)^{1/p}, \tag{13.1}$$

where $\mathbf{f} = (f_1, \dots, f_m)$, $p, q \in [1, \infty]$ (with the usual meaning if $p = \infty$ and/or $q = \infty$). If $p = q$, the $A(p, q)$ and $B(p, q)$ norms are identical. As such we will not consider this case. For $1 < p, q < \infty$, the $B(p, q)$-norm is strictly convex and every finite dimensional subspace is therefore a unicity space. This leaves 6 general cases for consideration.

The general form of the dual space is similar to that in the $A(p, q)$ case. For $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$, we have $B^*(p, q) = B(p', q')$, where $1/p + 1/p' = 1/q + 1/q' = 1$. That is, we may identify the dual of $B(p, q)$ as vectors

$$\mathbf{h} = (h_1, \dots, h_m),$$

with norm

$$\|\mathbf{h}\|_{B(p',q')} = \left(\int_D \left(\sum_{i=1}^m |f_i(x)|^{q'} \right)^{p'/q'} dv(x) \right)^{1/p'} \tag{13.2}$$

with the usual undestanding if $p' = \infty$ and/or $q' = \infty$. In these cases, it is also possible to determine the extreme points of the unit ball of the dual space of $B(p, q)$, and if needed, we will do so.

In the above paragraph, we did not mention the case where $p=\infty$. This is a more difficult problem, but the determination of the dual and the identification of the extreme points of the unit ball of the dual is a special case of a result obtained by Singer, see, e.g., [33, II.1.4]. For $p=\infty$ we always consider D as a compact Hausdorff set and restrict ourselves to $f_i \in C(D)$ (rather than $f_i \in L^\infty(D)$). The dual of $B(\infty, q)$ (under the above assumptions) may be identified with

$$\mu = (\mu_1, ..., \mu_m),$$

where each $\mu_i \in C^*(D)$ (i.e., regular Borel measures). To find the norm of μ, we set, for any Borel set B in D,

$$\|\mu(B)\|_{q'} = \left(\sum_{i=1}^{m} |\mu_i(B)|^{q'} \right)^{1/q'}.$$

The norm on μ is given by

$$\|\mu\|_{TV(q')} = \sup \sum_{j=1}^{r} \|\mu(D_j)\|_{q'},$$

where the supremum is taken over all finite partitions of D into pairwise disjoint Borel sets $\{D_j\}_{j=1}^r$. The vector μ operates on $\mathbf{f}$ in the simple form

$$\mu(\mathbf{f}) = \sum_{i=1}^{m} \int_D f_i(x)\, d\mu_i(x).$$

The extreme points of the unit ball of the dual space are the functionals of the form

$$\mu^* = (\mu_1^*, ..., \mu_m^*),$$

where $\mu_i^* = a_i^* \delta_{x^*}$ for some $a_i^* \in \mathbb{R}$, $x^* \in D$ (where δ_{x^*} represents point evaluation (Dirac–Delta measure) at x^*), and $(a_1^*, ..., a_m^*)$ is an extreme point of the unit ball in $l_{q'}^m$.

It is also possible to give a formula for $\tau_+^{B(p,q)}(\mathbf{f}, \mathbf{g})$. If $Y = l_q^m$, and $X = L^p(D, v)$ for $1 \leqslant p < \infty$, while $X = C(D)$ for $p = \infty$ then

$$\tau_+^{B(p,q)}(\mathbf{f}, \mathbf{g}) = \tau_+^X(\|\mathbf{f}(x)\|_Y, \tau_+^Y(\mathbf{f}(x), \mathbf{g}(x)))$$

(see Ioffe and Levin [13, p. 41] for the case where $1 \leqslant p < \infty$, and [13, Sect. 6] for the special case of $X = C(D)$).

14. $B(1, q)$, $1 < q < \infty$

Many of the results of this section are known. We include these results for completeness and because we regard this work as at least partially a survey. Let X be a norm on $\mathbb{R}^m$. We assume that for each $\mathbf{f} \in Y$,

$$\mathbf{f}(x) = (f_1(x), \ldots, f_m(x)), \qquad x \in D$$

the function $\|\mathbf{f}(x)\|_X$, as a function of x, is ν-integrable over D. (For convenience we assume that ν is σ-finite.) We set

$$\|\mathbf{f}\|_Y = \int_D \|\mathbf{f}(x)\|_X \, d\nu(x).$$

With no additional assumptions we obtain (see, e.g., Rozema [31, p. 592]) the following analogue of Theorem 3.4.

THEOREM 14.1. *Let U be a finite dimensional subspace of Y. Then $\mathbf{u}^*$ is a best approximant to $\mathbf{f}$ from U if and only if*

$$-\int_{D \setminus Z(\mathbf{f}-\mathbf{u}^*)} \tau_+^X(\mathbf{f}-\mathbf{u}^*, \mathbf{u})(x) \, d\nu(x) \leqslant \int_{Z(\mathbf{f}-\mathbf{u}^*)} \|\mathbf{u}(x)\|_X \, d\nu(x) \tag{14.1}$$

for all $\mathbf{u} \in U$.

If X is smooth, then $\tau_+(\mathbf{f}, \mathbf{g}) = \tau_-(\mathbf{f}, \mathbf{g})$ for all $\mathbf{f}, \mathbf{g}$ $(\mathbf{f} \neq \mathbf{0})$, and the common value is given by the unique norm one linear functional on X attaining its norm on $\mathbf{f}$, applied to $\mathbf{g}$. As such, we have:

PROPOSITION 14.2. *Assume X is smooth. Then $\mathbf{u}^*$ is a best approximant to $\mathbf{f}$ from the finite dimensional subspace U of Y if and only if*

$$\left| \int_{D \setminus Z(\mathbf{f}-\mathbf{u}^*)} \phi_{(\mathbf{f}-\mathbf{u}^*)(x)}(\mathbf{u}(x)) \, d\nu(x) \right| \leqslant \int_{Z(\mathbf{f}+\mathbf{u}^*)} \|\mathbf{u}(x)\|_X \, d\nu(x) \tag{14.2}$$

for all $\mathbf{u} \in U$, where $\phi_{(\mathbf{f}-\mathbf{u}^*)(x)}$ is the unique norm one linear functional on X attaining its norm at $(\mathbf{f}-\mathbf{u}^*)(x)$. Equivalently to (14.2), we have that there exist, for each $x \in Z(\mathbf{f}-\mathbf{u}^*)$, $\phi_x \in X^*$ of norm at most one such that

$$\int_{D \setminus Z(\mathbf{f}-\mathbf{u}^*)} \phi_{(\mathbf{f}-\mathbf{u}^*)(x)}(\mathbf{u}(x)) \, d\nu(x) + \int_{Z(\mathbf{f}-\mathbf{u}^*)} \phi_x(\mathbf{u}(x)) \, d\nu(x) = 0 \tag{14.3}$$

for all $\mathbf{u} \in U$.

In the case under consideration, namely $X = l_q^m$, $1 < q < \infty$, (14.2) translates into

$$\left| \int_{D \setminus Z(\mathbf{f}-\mathbf{u}^*)} \frac{\sum_{i=1}^m |(f_i - u_i^*)(x)|^{q-1} \operatorname{sgn}((f_i - u_i^*)(x))\, u_i(x)}{(\sum_{j=1}^m |(f_j - u_j^*)(x)|^q)^{1/q'}} \, dv(x) \right|$$
$$\leqslant \int_{Z(\mathbf{f}-\mathbf{u}^*)} \left(\sum_{i=1}^m |u_i(x)|^q \right)^{1/q} dv(x) \tag{14.4}$$

for all $\mathbf{u} \in U$. Equation (14.3) may be restated as follows: There exists for each $x \in Z(\mathbf{f} - \mathbf{u}^*)$, a vector $(\phi_1(x), ..., \phi_m(x))$ satisfying

$$\left(\sum_{i=1}^m |\phi_i(x)|^{q'} \right)^{1/q'} \leqslant 1$$

for all $x \in Z(\mathbf{f} - \mathbf{u}^*)$ and such that

$$\sum_{i=1}^m \left[\int_{D \setminus Z(\mathbf{f}-\mathbf{u}^*)} \frac{|(f_i - u_i^*)(x)|^{q-1} \operatorname{sgn}((f_i - u_i^*)(x))\, u_i(x)}{(\sum_{j=1}^m |(f_j - u_j^*)(x)|^q)^{1/q'}} \, dv(x) \right.$$
$$\left. + \int_{Z(\mathbf{f}-\mathbf{u}^*)} \phi_i(x)\, u_i(x)\, dv(x) \right] = 0 \tag{14.5}$$

for all $\mathbf{u} \in U$. If the measure v is non-atomic, it is permissible to assume that

$$\left(\sum_{i=1}^m |\phi_i(x)|^{q'} \right)^{1/q'} = 1$$

for all $x \in Z(\mathbf{f} - \mathbf{u}^*)$. In what follows we always assume that v is non-atomic.

We restrict ourselves, in general for convenience only, to

$$Y_q = \{ \mathbf{f} : \mathbf{f}(x) = (f_1(x), ..., f_m(x)), x \in D \},$$

where $1 < q < \infty$, and

$$\|\mathbf{f}\|_{Y_q} = \int_D \left(\sum_{i=1}^m |f_i(x)|^q \right)^{1/q} dv(x)$$

(that is, the functions $\mathbf{f}$ are such that $(\sum_{i=1}^m |f_i(x)|^q)^{1/q} \in L^1(D, v)$) and v is a non-atomic σ-finite measure.

This next result, analogous to Theorem 3.5, is a special case of a result of Rozema [31, Theorem 2.1].

THEOREM 14.3. *Let U be a finite dimensional subspace of Y_q, where q and v are as above. Then U is not a unicity space for Y_q.*

Proof. Choose any $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$. Let $h \in L^\infty(D, \nu)$ satisfy

(1) $|h(x)| = 1$ all $x \in D$

(2) $$\sum_{i=1}^{m} \int_{D \setminus Z(\mathbf{u}^*)} \frac{|u_i^*(x)|^{q-1} \operatorname{sgn}(u_i^*(x))\, u_i(x)\, h(x)}{(\sum_{j=1}^{m} |u_j^*(x)|^q)^{1/q'}}\, d\nu(x) = 0 \quad \text{all} \quad \mathbf{u} \in U.$$

Such an h exists because ν is non-atomic and σ-finite.

Set $\mathbf{f}(x) = h(x)\,\mathbf{u}^*(x)$, i.e., $f_i(x) = h(x)\, u_i(x)$ for $i = 1, ..., m$, $x \in D$. Thus $\mathbf{f} \in Y_q$, $Z(\mathbf{f}) = Z(\mathbf{u}^*)$, $\operatorname{sgn}(f_i(x)) = h(x) \operatorname{sgn}(u_i^*(x))$, and $|f_i(x)| = |u_i(x)|$ for all $i = 1, ..., m$, and $x \in D$. From (2),

$$\sum_{i=1}^{m} \int_{D \setminus Z(\mathbf{f})} \frac{|f_i(x)|^{q-1} \operatorname{sgn}(f_i(x))\, u_i(x)}{(\sum_{j=1}^{m} |f_j(x)|^q)^{1/q'}}\, d\nu(x) = 0 \tag{14.6}$$

for all $\mathbf{u} \in U$. Thus from (14.4) we have that $\mathbf{0}$ is a best approximant to $\mathbf{f}$ from U.

For $|\alpha| < 1$, $Z(\mathbf{f} - \alpha \mathbf{u}^*) = Z(\mathbf{f})$, and

$$\frac{|(f_i - \alpha u_i^*)(x)|^{q-1} \operatorname{sgn}((f_i - \alpha u_i^*)(x))\, u_i(x)}{(\sum_{j=1}^{m} |(f_j - \alpha u_j^*)(x)|^q)^{1/q'}} = \frac{|f_i(x)|^{q-1} \operatorname{sgn}(f_i(x))\, u_i(x)}{(\sum_{j=1}^{m} |f_j(x)|^q)^{1/q'}}$$

for each $x \in D$. Substituting $\mathbf{f} - \alpha \mathbf{u}^*$ in place of $\mathbf{f}$ in (14.6), we get from (14.4) that $\alpha \mathbf{u}^*$ is a best approximant to $\mathbf{f}$ from U for all $|\alpha| \leqslant 1$. ∎

As in previous sections, we restrict ourselves to $D = K \subset \mathbb{R}^d$, compact, satisfying $K = \overline{\operatorname{int} K}$, and measures $\mu \in \mathscr{A}$. By CY_q we mean the restriction of Y_q to $\mathbf{f} = (f_1, ..., f_m)$ such that $f_i \in C(K)$, $i = 1, ..., m$. There is a dependence on μ which is to be understood. Under these assumptions we have analogues of Theorems 3.6 and 3.7. We list one after the other and prove them simultaneously.

THEOREM 14.4. *The finite dimensional subspace U of CY_q is a unicity space if and only if there does not exist an $h \in L^\infty(K, \mu)$ and a $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ such that*

(1) $|h(x)| = 1$ *all* $x \in K$

(2) $h(x)\, u_i^*(x) \in C(K)$, $i = 1, ..., m$

(3) $$\left| \int_{K \setminus Z(\mathbf{u}^*)} \frac{\sum_{i=1}^{m} |u_i^*(x)|^{q-1} \operatorname{sgn}(u_i^*(x))\, u_i(x)\, h(x)}{(\sum_{j=1}^{m} |u_j^*(x)|^q)^{1/q'}}\, d\mu(x) \right| \leqslant \int_{Z(\mathbf{u}^*)} \left(\sum_{i=1}^{m} |u_i(x)|^q \right)^{1/q} d\mu(x) \quad \textit{all} \quad \mathbf{u} \in U.$$

Given U as above, we define

$$U^* = \{\mathbf{g} : \mathbf{g} \in CY_q, \mathbf{g}(x) = h(x)\,\mathbf{u}(x), \mathbf{u} \in U, |h(x)| = 1, \text{ all } x \in K\}$$

(analogous to the definition of U^* in Theorem 3.7).

THEOREM 14.5 (Kroó [21, Theorem 1]). *The finite dimensional subspace U of CY_q is a unicity space for CY_q if and only if $\mathbf{0}$ is not a best approximant from U to any $\mathbf{g} \in U^* \setminus \{\mathbf{0}\}$.*

Proof of Theorems 14.4 *and* 14.5. (a) Assume that U is not a unicity space. Let $\mathbf{f} \in CY_q$, and $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ be such that $\pm \mathbf{u}^*$ are best approximants of $\mathbf{f}$ from U. For each $x \in K$ and $i \in \{1, ..., m\}$

$$2\,|f_i(x)| \leqslant |(f_i - u_i^*)(x)| + |(f_i + u_i^*)(x)| \tag{14.7}$$

and thus

$$\begin{aligned} 2&\left(\sum_{i=1}^{m} |f_i(x)|^q\right)^{1/q} \\ &\leqslant \left(\sum_{i=1}^{m} |(f_i - u_i^*)(x)|^q\right)^{1/q} + \left(\sum_{i=1}^{m} |(f_i + u_i^*)(x)|^q\right)^{1/q}. \end{aligned} \tag{14.8}$$

Since

$$\begin{aligned} 2&\int_K \left(\sum_{i=1}^{m} |f_i(x)|^q\right)^{1/q} d\mu(x) \\ &= \int_K \left(\sum_{i=1}^{m} |(f_i - u_i^*)(x)|^q\right)^{1/q} d\mu(x) + \int_K \left(\sum_{i=1}^{m} |(f_i + u_i^*)(x)|^q\right)^{1/q} d\mu(x) \end{aligned}$$

it follows that equality holds in (14.7) and (14.8) for all $x \in K$ and $i \in \{1, ..., m\}$. As such $(1 < q < \infty)$, for each $x \in K$, either $(f_i + u_i^*)(x) = 0$, $i = 1, ..., m$, or there exists a constant $c(x) \geqslant 0$ such that

$$(f_i - u_i^*)(x) = c(x)((f_i + u_i^*)(x)),$$

$i = 1, ..., m$. These two options translate into the existence of $\gamma(x)$ satisfying $|\gamma(x)| \leqslant 1$, and

$$u_i^*(x) = \gamma(x) f_i(x), \qquad x \in K, i = 1, ..., m. \tag{14.9}$$

In particular, we obtain $Z(\mathbf{f}) \subseteq Z(\mathbf{u}^*)$ (which also follows from equality in (14.7)). Since $u_i^*, f_i \in C(K)$ for each i, we also have that γ is continuous on $K \setminus Z(\mathbf{f})$.

Set

$$\mathbf{g}(x) = (\operatorname{sgn} \gamma(x))\, \mathbf{u}^*(x).$$

From (14.9),

$$\mathbf{g}(x) = (\operatorname{sgn} \gamma(x))\, \mathbf{u}^*(x) = (\operatorname{sgn} \gamma(x))\, \gamma(x)\, \mathbf{f}(x) = |\gamma(x)|\, \mathbf{f}(x).$$

Thus $\mathbf{g} \in CY_q$. From (14.9), $\gamma(x) \neq 0$ for $x \in K \setminus Z(\mathbf{u}^*)$. Thus $\mathbf{g} \in U^*$ and $Z(\mathbf{g}) = Z(\mathbf{u}^*)$. As is easily checked, for each $x \in K \setminus Z(\mathbf{g})$ and $i = 1, \ldots, m$,

$$\frac{|g_i(x)|^{q-1} \operatorname{sgn}(g_i(x))}{(\sum_{j=1}^m |g_j(x)|^q)^{1/q'}} = \frac{|f_i(x)|^{q-1} \operatorname{sgn}(f_i(x))}{(\sum_{j=1}^m |f_j(x)|^q)^{1/q'}}.$$

Since, by assumption, $\mathbf{0}$ is a best approximant to $\mathbf{f}$ from U, it now follows from (14.4), for example, that $\mathbf{0}$ is a best approximant to $\mathbf{g}$ from U.

(b) Assume $\mathbf{g} \in U^* \setminus \{\mathbf{0}\}$ and $\mathbf{0}$ is a best approximant to $\mathbf{g}$ from U. Since $\mathbf{g} \in U^* \setminus \{\mathbf{0}\}$, we have

$$\mathbf{g}(x) = h(x)\, \mathbf{u}^*(x),$$

where (1) and (2) of Theorem 14.4 hold for this h and $\mathbf{u}^*$. Now $Z(\mathbf{g}) = Z(\mathbf{u}^*)$ and for $x \in K \setminus Z(\mathbf{g})$ and $i \in \{1, \ldots, m\}$

$$\frac{|g_i(x)|^{q-1} \operatorname{sgn}(g_i(x))}{(\sum_{j=1}^m |g_j(x)|^q)^{1/q'}} = \frac{|u_i^*(x)|^{q-1} \operatorname{sgn}(u_i^*(x))\, h(x)}{(\sum_{j=1}^m |u_j^*(x)|^q)^{1/q'}}.$$

Substituting this equality in (14.4) where we use the fact that $\mathbf{0}$ is a best approximant to $\mathbf{g}$ from U, we obtain (3) of Theorem 14.4.

(c) Assume there exists an $h \in L^\infty(K, \mu)$ and $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ satisfying (1), (2), and (3) as in Theorem 14.4. From the method of proof of Theorem 14.3, we have that $\mathbf{f}(x) = h(x)\, \mathbf{u}^*(x)$ is such that $\alpha \mathbf{u}^*$ is a best approximant to $\mathbf{f}$ from U for all $|\alpha| \leqslant 1$. From (2), $\mathbf{f} \in CY_q$ ($\mathbf{f} \in U^*$). Thus U is not a unicity space for CY_q.

The above (a), (b), and (c) prove Theorems 14.4 and 14.5. ∎

The conditions of Theorems 14.4 and 14.5 are generally difficult to verify. Let us consider some simpler examples.

EXAMPLE 1. There is one simple case where U is a unicity space for CY_q for every $q \in (1, \infty)$ and $\mu \in \mathscr{A}$, and that is when $K \setminus Z(\mathbf{u})$ is connected for every $\mathbf{u} \in U \setminus \{\mathbf{0}\}$. Such a situation may well occur if K is connected and $n \leqslant m$. For then we may have $Z(\mathbf{u}) = \varnothing$ for every $\mathbf{u} \in U \setminus \{\mathbf{0}\}$ (in this regard, see Section 16). If $K \setminus Z(\mathbf{u})$ is connected for every $\mathbf{u} \in U \setminus \{\mathbf{0}\}$, then $U^* = U$ and an application of Theorem 14.5 easily proves the unicity property.

EXAMPLE 2. dim $U = 1$. In this case where $U = \text{span}\{\mathbf{u}^*\}$, the conditions (1), (2), and (3) of Theorem 14.4 reduce to the existence of $h \in L^\infty(K, \mu)$ satisfying

(1′) $|h(x)| = 1$ all $x \in K$

(2′) $h(x)\, u_i^*(x) \in C(K)$, $i = 1, \ldots, m$

(3′) $\int_K (\sum_{i=1}^m |u_i^*(x)|^q)^{1/q} h(x)\, d\mu(x) = 0$.

EXAMPLE 3. Simultaneous Approximation. We assume that $\tilde{U}$ is an n-dimensional subspace of $C(K)$ and

$$U\{\mathbf{u} = (u, \ldots, u) : u \in \tilde{U}\}.$$

PROPOSITION 14.6. *U is a unicity space for CY_q if and only if $\tilde{U}$ is a unicity space for $C_1(K, \mu)$.*

We can prove this result using either Theorem 14.4 or 14.5. Note that Theorem 14.5 essentially says that the unicity space property is checked by verifying it on a set of "test functions," namely the functions in U^*. All functions in U^* inherit from U the property that their m components are all the same. But for all such functions, their Y_q norm is just $m^{1/q}$ times the $L^1(K, \mu)$ norm of any component. Thus Proposition 14.6 holds.

EXAMPLE 4. Tensor Product. Assume $U = U^1 \oplus \cdots \oplus U^m$. As a partial result we have the following.

PROPOSITION 14.7. *If U^j is not a unicity space for $C_1(K, \mu)$ for some $j \in \{1, \ldots, m\}$, then U is not a unicity space for CY_q.*

Proof. Since U^j is not a unicity space, there exists a $u_j^* \in U^j \setminus \{0\}$ and an $f_j \in C(K)$ such that $\pm u_j^*$ are best approximants to f_j from U^j in the $L^1(K, \mu)$ norm. Set $\mathbf{f} = (0, \ldots, 0, f_j, 0, \ldots, 0)$, and $\mathbf{u}^* = (0, \ldots, 0, u_j^*, 0, \ldots, 0)$. Then $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ and it is easily checked that $\pm \mathbf{u}^*$ are best CY_q approximants to $\mathbf{f}$ from U. ∎

As noted many times in this paper, conditions of the type found in Theorems 14.4 and 14.5 are both difficult to verify and dependent on the particular choice of the measure $\mu \in \mathscr{A}$. As such it is natural to try to find conditions on the subspace U which guarantee that it is a unicity space for all $\mu \in \mathscr{A}$. This condition we call *Property* A_q.

DEFINITION. The finite dimensional subspace U of CY_q is said to satisfy Property A_q if given any $\mathbf{g} \in U^* \setminus \{\mathbf{0}\}$ there exists a $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ satisfying

(1) $\mathbf{u}^* = \mathbf{0}$ (Lebesgue) a.e. on $Z(\mathbf{g})$

(2) $\sum_{i=1}^m |g_i(x)|^{q-1} \operatorname{sgn}(g_i(x))\, u_i^*(x) \geqslant 0$ for all $x \in K$ and it is strictly positive on a set of positive (Lebesgue) measure.

Remark. For other related definitions of Property A_q, see Section 3.

THEOREM 14.8 (Kroó [21, Theorems 2 and 3]). *The finite dimensional subspace U of CY_q is a unicity space for CY_q for all measures $\mu \in \mathscr{A}$ if and only if U satisfies Property A_q.*

Remark. Kroó in [21] proves the above result with any smooth strictly convex space in place of l_q^m. In fact, as noted by Kroó, one direction in the proof uses the smoothness and the other direction the strict convexity.

Proof. ($\Leftarrow$) Assume U is not a unicity space for some $\mu \in \mathscr{A}$. From Theorem 14.5 there exists a $\mathbf{g} \in U^* \setminus \{\mathbf{0}\}$ such that $\mathbf{0}$ is a best approximant to $\mathbf{g}$ from U in the $Y_q(\mu)$ norm. Thus, from (14.4)

$$\left| \int_{K \setminus Z(\mathbf{g})} \frac{\sum_{i=1}^m |g_i(x)|^{q-1} \operatorname{sgn}(g_i(x))\, u_i(x)}{\left(\sum_{j=1}^m |g_j(x)|^q\right)^{1/q'}}\, d\mu(x) \right| \leqslant \int_{Z(\mathbf{g})} \left(\sum_{i=1}^m |u_i(x)|^q \right)^{1/q} d\mu(x)$$

for all $\mathbf{u} \in U$. Let $\mathbf{u}^*$ be as given by Property A_q satisfying (1) and (2) in the definition thereof. From (1),

$$\int_{Z(\mathbf{g})} \left(\sum_{i=1}^m |u_i^*(x)|^q \right)^{1/q} d\mu(x) = 0.$$

From (2),

$$\int_{K \setminus Z(\mathbf{g})} \frac{\sum_{i=1}^m |g_i(x)|^{q-1} \operatorname{sgn}(g_i(x))\, u_i^*(x)}{\left(\sum_{j=1}^m |g_j(x)|^q\right)^{1/q'}}\, d\mu(x) > 0.$$

This is a contradiction.

($\Rightarrow$) Assume Property A_q does not hold. That is, there exists a $\mathbf{g} \in U^* \setminus \{\mathbf{0}\}$ such that if $\mathbf{v} \in U$ satisfies

(1′) $\mathbf{v} = \mathbf{0}$ (Lebesgue) a.e. on $\mathbf{Z}(\mathbf{g})$

(2′) $\sum_{i=1}^m |g_i(x)|^{q-1} \operatorname{sgn}(g_i(x))\, v_i(x) \geqslant 0$ for all $x \in K$, then equality holds identically in (2′). Since $\mathbf{g} \in U^*$, we have $\mathbf{g} = h\tilde{\mathbf{u}}$ for some h satisfying $|h(x)| = 1$ all $x \in K$, and $\tilde{\mathbf{u}} \in U \backslash \{\mathbf{0}\}$.

Set

$$U_1 = \{\mathbf{u} : \mathbf{u} \in U, \mathbf{u} = \mathbf{0} \text{ a.e. on } \mathbf{Z}(\mathbf{g})\}.$$

U_1 is a linear subspace of U of dimension k, $1 \leqslant k \leqslant n$ (since $\tilde{\mathbf{u}} \in U_1$). Let

$$U_1 = \operatorname{span}\{\mathbf{u}^1, ..., \mathbf{u}^k\}.$$

From the above, if $\mathbf{v} \in U_1$ satisfies (2′) then equality holds in (2′). Thus it may be shown (see Pinkus [29, p. 61]) that there exists a measure $\mu \in \mathscr{A}$ (defined on $K \backslash Z(\mathbf{g})$) such that

$$\int_{K \backslash Z(\mathbf{g})} \frac{\sum_{i=1}^m |g_i(x)|^{q-1} \operatorname{sgn}(g_i(x))\, u_i(x)}{(\sum_{j=1}^m |g_j(x)|^q)^{1/q'}}\, d\mu(x) = 0$$

for all $\mathbf{u} \in U_1$.

Let

$$U_2 = \operatorname{span}\{\mathbf{u}^{k+1}, ..., \mathbf{u}^n\},$$

where $U = \operatorname{span}\{\mathbf{u}^1, ..., \mathbf{u}^n\}$. Set

$$\eta_1(\mathbf{u}) = \int_{Z(\mathbf{g})} \left(\sum_{i=1}^m |u_i(x)|^q \right)^{1/q} dx$$

and

$$\eta_2(\mathbf{u}) = \max_{x \in K} \left(\sum_{i=1}^m |u_i(x)|^q \right)^{1/q}.$$

The subspace U_2 is "linearly independent over $Z(\mathbf{g})$." That is, if $\mathbf{u} \in U_2$ is such that $\mathbf{u} = \mathbf{0}$ a.e. on $Z(\mathbf{g})$, then $\mathbf{u} = \mathbf{0}$ on all K. Now $\eta_2(\mathbf{u})$ is a norm on U_2. From the above $\eta_1(\mathbf{u})$ is also a norm on U_2. Since U_2 is finite dimensional, this implies the existence of a $c > 0$ such that

$$\eta_2(\mathbf{u}) \leqslant c\eta_1(\mathbf{u})$$

for all $\mathbf{u} \in U_2$.

Let μ be as above, defined on $K \backslash Z(\mathbf{g})$, and $\mu(K \backslash Z(\mathbf{g})) = m$. Let $\tilde{\mu} \in \mathscr{A}$ be defined to be equal to μ on $K \backslash Z(\mathbf{g})$ and to be mc times Lebesgue measure on $Z(\mathbf{g})$.

Given $\mathbf{u} \in U$, we write $\mathbf{u} = \mathbf{u}^1 + \mathbf{u}^2$ where $\mathbf{u}^k \in U_k$, $k = 1, 2$. Now,

$$\begin{aligned}
&\left| \int_{K \setminus Z(\mathbf{g})} \frac{\sum_{i=1}^m |g_i(x)|^{q-1} \operatorname{sgn}(g_i(x))(u_i^1(x) + u_i^2(x))}{(\sum_{j=1}^m |g_j(x)|^q)^{1/q'}} \, d\tilde{\mu}(x) \right| \\
&\quad = \left| \int_{K \setminus Z(\mathbf{g})} \frac{\sum_{i=1}^m |g_i(x)|^{q-1} \operatorname{sgn}(g_i(x))\, u_i^2(x)}{(\sum_{j=1}^m |g_j(x)|^q)^{1/q'}} \, d\mu(x) \right| \\
&\quad \leqslant \int_{K \setminus Z(\mathbf{g})} \left(\sum_{i=1}^m |u_i^2(x)|^q \right)^{1/q} d\mu(x) \\
&\quad \leqslant m\eta_2(\mathbf{u}^2) \\
&\quad \leqslant cm\eta_1(\mathbf{u}^2) \\
&\quad = \int_{Z(\mathbf{g})} \left(\sum_{i=1}^m |u_i^2(x)|^q \right)^{1/q} d\tilde{\mu}(x) \\
&\quad = \int_{Z(\mathbf{g})} \left(\sum_{i=1}^m |u_i(x)|^q \right)^{1/q} d\tilde{\mu}(x).
\end{aligned}$$

It therefore follows from (14.4) and Theorem 14.5 that U is not a unicity space for CY_q with respect to the measure $\tilde{\mu} \in \mathscr{A}$. ∎

We now turn to our three main examples to see conditions under which they satisfy Property A_q.

EXAMPLE 1. $\dim U = 1$. It is easily seen that $U = \operatorname{span}\{\mathbf{u}^*\}$ satisfies Property A_q if and only if $K \setminus Z(\mathbf{u}^*)$ is connected.

EXAMPLE 2. Simultaneous Approximation. From Proposition 14.6 we have that U satisfies Property A_q if and only if $\tilde{U}$ satisfies Property A, i.e., $\tilde{U}$ is a unicity space for $C_1(K, \mu)$ for all $\mu \in \mathscr{A}$.

EXAMPLE 3. Tensor Product. Assume $U = U^1 \oplus \cdots \oplus U^m$.

PROPOSITION 14.9. *U satisfies Property A_q if and only if each of $U^1, ..., U^m$ satisfies Property* A.

Proof. ($\Rightarrow$) This follows from Proposition 14.7.

($\Leftarrow$) Assume each of $U^1, ..., U^m$ satisfy Property A. Let $\mathbf{g} \in U^*$, $\mathbf{g} = (g_1, ..., g_m)$. If $g_i \neq 0$, then $g_i \in (U^i)^*$. Thus there exists a $u_i \in (U^i)^* \setminus \{0\}$ satisfying

(1′) $u_i = 0$ a.e. on $Z(g_i)$

(2′) $g_i u_i \geqslant 0$ on K

(and since $u_i \neq 0$, strict inequality holds in (2′) on a set of positive Lebesgue measure). Set $\mathbf{u} = (u_1, ..., u_m)$. Then $\mathbf{u} \in U \setminus \{\mathbf{0}\}$ and $\mathbf{u}$ satisfies (1) and (2) in the definition of Property A_q. ∎

15. $B(p, 1)$, $1 < p < \infty$

D is a set and ν a positive σ-finite measure. For $1 < p < \infty$, we let Y_p denote the set of functions $\mathbf{f} = (f_1, ..., f_m)$ with norm

$$\|\mathbf{f}\|_{Y_p} = \left(\int_D \left(\sum_{i=1}^m |f_i(x)|\right)^p d\nu(x)\right)^{1/p}.$$

As noted in Section 13, the dual space may be identified with $B(p', \infty)$. That is, the set of $\mathbf{h} = (h_1, ..., h_m)$ normed by

$$\|\mathbf{h}\|_{Y_{p'}} = \left(\int_D [\max_{i=1,...,m} |h_i(x)|]^{p'} d\nu(x)\right)^{1/p'}.$$

Characterization of best approximants is readily attained from any of the various results and techniques at our disposal.

THEOREM 15.1. *Let U be a finite dimensional subspace of Y_p. Then $\mathbf{u}^*$ is a best approximant to $\mathbf{f}$ from U if and only if*

$$\left|\sum_{i=1}^m \int_D \left(\sum_{j=1}^m |(f_j - u_j^*)(x)|\right)^{p-1} \operatorname{sgn}((f_i - u_i^*)(x))\, u_i(x)\, d\nu(x)\right|$$
$$\leqslant \sum_{i=1}^m \int_{Z(f_i - u_i^*)} \left(\sum_{j=1}^m |(f_j - u_j^*)(x)|\right)^{p-1} |u_i(x)|\, d\nu(x) \qquad (15.1)$$

for all $\mathbf{u} \in U$. Equivalently, we have that there exist $h_i \in L^\infty(Z(f_i - u_i^*))$, $i = 1, ..., m$, satisfying $\|h_i\|_\infty \leqslant 1$ and

$$0 = \sum_{i=1}^m \left[\int_{N(f_i - u_i^*)} \left(\sum_{j=1}^m |(f_j - u_j^*)(x)|\right)^{p-1} \right.$$
$$\times \operatorname{sgn}((f_i - u_i^*)(x))\, u_i(x)\, d\nu(x)$$
$$\left. + \int_{Z(f_i - u_i^*)} \left(\sum_{j=1}^m |(f_j - u_j^*)(x)|\right)^{p-1} h_i(x)\, u_i(x)\, d\nu(x)\right] \qquad (15.2)$$

for all $\mathbf{u} \in U$.

If, in addition, the measure ν is non-atomic, then we may assume that in (15.2) $|h_i(x)| = 1$ for all $x \in Z(f_i - u_i^*)$ and $i = 1, ..., m$.

In what follows we assume that ν is a non-atomic finite measure on D. We can then state a simple elegant criteria for unicity.

THEOREM 15.2. *Let U be a finite dimensional subspace of Y_p $(1 < p < \infty)$. Then U is a unicity space if and only if there does not exist a $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ and $\mathbf{h} = (h_1, ..., h_m)$ $(\in Y_p^*)$ such that for all $x \in D$*

(1) $|h_i(x)| = 1$, *all* $i = 1, ..., m$,

(2) $\sum_{i=1}^m h_i(x)\, u_i^*(x) = 0$.

Proof. $(\Leftarrow)$ Assume U is not a unicity space for Y_p. Let $\mathbf{f} \in Y_p$ be such that $\pm \mathbf{u}^*$ are best approximants to $\mathbf{f}$ from U for some $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$. Now

$$2\, |f_i(x)| \leqslant |(f_i - u_i^*)(x)| + |(f_i + u_i^*)(x)| \tag{15.3}$$

for each i and x which implies

$$2 \sum_{i=1}^m |f_i(x)| \leqslant \sum_{i=1}^m |(f_i - u_i^*)(x)| + \sum_{i=1}^m |(f_i + u_i^*)(x)|$$

and thus

$$2\, \|\mathbf{f}\|_{Y_p} \leqslant \|\mathbf{f} - \mathbf{u}^*\|_{Y_p} + \|\mathbf{f} + \mathbf{u}^*\|_{Y_p}. \tag{15.4}$$

But equality must hold in (15.4). Since $L^p(D, \nu)$ is a strictly convex norm, we have that

$$\sum_{i=1}^m |(f_i - u_i^*)(x)| = c \sum_{i=1}^m |(f_i + u_i^*)(x)|, \qquad \nu \text{ a.e.}$$

for some constant $c \geqslant 0$ (or the right-hand side is identically zero). But

$$\|\mathbf{f} - \mathbf{u}^*\|_{Y_p} = \|\mathbf{f} + \mathbf{u}^*\|_{Y_p} > 0.$$

Thus $c = 1$.

From equality in (15.3), we obtain

$$|f_i(x)| \geqslant |u_i^*(x)| \qquad \nu \text{ a.e.}$$

for each $i = 1, ..., m$. (Thus $Z(f_i) \subseteq Z(u_i^*)$ ν a.e., $i = 1, ..., m$.) As such, ν a.e.

$$\begin{aligned}\sum_{i=1}^m |(f_i \pm u_i^*)(x)| &= \sum_{i=1}^m \operatorname{sgn}(f_i(x))(f_i \pm u_i^*)(x)\\ &= \sum_{i=1}^m |f_i(x)| \pm \sum_{i=1}^m \operatorname{sgn}(f_i(x))\, u_i^*(x).\end{aligned}$$

Thus we get

$$\sum_{i=1}^{m} \operatorname{sgn}(f_i(x))\, u_i^*(x) = 0 \qquad \nu \text{ a.e.}$$

For each i, let $h_i \in L^\infty(D, \nu)$ be chosen such that $|h_i(x)| = 1$ for all $x \in D$, and $h_i(x) = \operatorname{sgn} f_i(x)$ on $D \backslash Z(f_i)$, ν a.e. Since $Z(f_i) \subseteq Z(u_i^*)$, ν a.e., it follows that we can choose h_i so that (2) also holds.

($\Rightarrow$) Assume there exist $\mathbf{h}$ and $\mathbf{u}^* \in U \backslash \{\mathbf{0}\}$ satisfying (1) and (2). Since ν is a non-atomic finite measure and U is a finite dimensional subspace there exists an $\varepsilon \in L^\infty(D, \nu)$ satisfying

(a) $|\varepsilon(x)| = 1$ all $x \in D$

(b) $\int_D (\sum_{j=1}^m |u_j^*(x)|)^{p-1} (\sum_{i=1}^m h_i(x)\, u_i(x))\, \varepsilon(x)\, d\nu(x) = 0$ for all $\mathbf{u} \in U$.

Set $\mathbf{f} = (f_1, ..., f_m)$ where

$$f_i(x) = \varepsilon(x)\, h_i(x)\, |u_i^*(x)|, \qquad x \in D,\ i = 1, ..., m.$$

For $|\alpha| < 1$, we have from (2) and since $|f_i(x)| = |u_i^*(x)|$,

$$\begin{aligned}
\sum_{j=1}^{m} |(f_j - \alpha u_j^*(x)| &= \sum_{j=1}^{m} [|u_j^*(x)| - \alpha\varepsilon(x)\, h_j(x)\, u_j^*(x)] \\
&= \sum_{j=1}^{m} |u_j^*(x)| - \alpha\varepsilon(x) \sum_{j=1}^{m} h_j(x)\, u_j^*(x) \\
&= \sum_{j=1}^{m} |u_j^*(x)|.
\end{aligned}$$

In addition,

$$\operatorname{sgn}((f_i - \alpha u_i^*)(x)) = \varepsilon(x)\, h_i(x)$$

on $D \backslash Z(u_i^*)$.

Substituting in (15.2), where the $h_i(x)$ therein is taken to be the $\varepsilon(x)\, h_i(x)$ as above, it follows that $\alpha \mathbf{u}^*$ is a best Y_p approximant to $\mathbf{f}$ from U for every $|\alpha| \leqslant 1$. ∎

Remark. Conditions (1) and (2) of Theorem 15.2 are independent of $p \in (1, \infty)$ and the particular measure ν. This is not surprising. The lack of unicity is a consequence of the l_1^m norm and not the $L^p(D, \nu)$ norm. A reading of the proof shows that this result can be proved for more general norms rather than $L^p(D, \nu)$. Note also that if $m = 1$ there is, of course, nothing to prove as Y_p is then $L^p(D, \nu)$.

In the case of simultaneous approximation, it is easily seen from Theorem 15.2 that U is a unicity space if and only if m is odd.

If we restrict our attention to the set of continuous functions and consider the unicity problem restricted to this set (as we have done in previous sections), then the conditions for a unicity space become more complicated. However, such conditions can be given.

Finally we note that in place of the Y_p norm as defined herein, we could have considered the norm

$$\|\mathbf{f}\|_{Y_p(\mathbf{w})} = \left(\int_D \left(\sum_{i=1}^{m} w_i(x)\,|f_i(x)|\right)^p dv(x)\right)^{1/p},$$

where the w_i are some suitable positive functions (weights). In this case Theorem 15.2 holds, but condition (2) thereof is replaced by

$$(2') \quad \sum_{i=1}^{m} h_i(x)\, w_i(x)\, u_i^*(x) = 0$$

for all $x \in D$. Obviously U is a unicity space in the $Y_p(\mathbf{w})$ norm for every such $\mathbf{w}$ if for each $\mathbf{u} \in U$, on a set of positive v measure, the cardinality of the set

$$\{i : u_i(x) \neq 0\}$$

is exactly 1. Whether this condition is also necessary in order that U be a unicity space in the $Y_p(\mathbf{w})$ norm for all $\mathbf{w}$ seems to depend on a more explicit definition of the admissible $\mathbf{w}$.

16. $B(\infty, q)$, $1 < q < \infty$

We let D be a compact Hausdoff space and $C(D)$ denote the space of continuous real-valued functions defined on D. We set

$$\mathbf{f}(x) = (f_1(x), \ldots, f_m(x)),$$

where each $f_i \in C(D)$. We norm $\mathbf{f}$ by

$$\|\mathbf{f}\|_Y = \max_{x \in D} \|\mathbf{f}(x)\|_X,$$

where X is any norm on $\mathbb{R}^m$. In this section we review results first obtained by Zuhovitsky and Stechkin [36]. Firstly, however, we present a characterization of best approximants (based on results to be found in Singer [33]).

THEOREM 16.1. *Let U be an n-dimensional subspace of Y. Then $\mathbf{u}^*$ is a best approximant to $\mathbf{f}$ from U if and only if for some k, $1 \leqslant k \leqslant n+1$, there exist points $\{x_j\}_{j=1}^k$ in D, positive numbers $\{\lambda_j\}_{j=1}^k$, and extremal points $\{\mathbf{h}^j\}_{j=1}^k$ of the unit bal of X^*, such that*

(1) $\sum_{j=1}^k \sum_{i=1}^m \lambda_j h_i^j u_i(x_j) = 0$, *all* $\mathbf{u} \in U$

(2) $\sum_{i=1}^m h_i^j (f_i - u_i^*)(x_j) = \|(\mathbf{f} - \mathbf{u}^*)(x_j)\|_X = \|\mathbf{f} - \mathbf{u}^*\|_Y$, $j = 1, ..., k$.

Remark. If $X = l_q^m$, $1 < q < \infty$, then

$$h_i^j = \frac{|(f_i - u_i^*)(x_j)|^{q-1} \operatorname{sgn}((f_i - u_i^*)(x_j))}{\|(\mathbf{f} - \mathbf{u}^*)(x_j)\|_q^{q-1}}$$

for each i and j. In this case we can substitute this h_i^j in (1), deleting the denominator, and replace (2) by

(2′) $\|(\mathbf{f} - \mathbf{u}^*)(x_j)\|_q = \|\mathbf{f} - \mathbf{u}^*\|_Y, \qquad j = 1, ..., k.$

In what follows we need the following notation. We recall that

$$Z(\mathbf{g}) = \{x : \mathbf{g}(x) = \mathbf{0}\}.$$

By $|Z(\mathbf{g})|$ we mean the cardinality of the set $Z(\mathbf{g})$, i.e., the number of distinct zeros of $\mathbf{g}(x)$ (in D).

Assuming X is strictly convex, we have the following generalization of Haar's Theorem (Theorem 3.3).

THEOREM 16.2. *Assume X is a strictly convex norm on $\mathbb{R}^m$. The n-dimensional subspace U of Y is a unicity space if and only if*

(a) $|Z(\mathbf{u})| < n/m$ *for all* $\mathbf{u} \in U \setminus \{\mathbf{0}\}$

(b) *if $k = [n/m]$, then for every choice of $x_1, ..., x_k$ (distinct) in D, and $\alpha^1, ..., \alpha^k \in \mathbb{R}^m$, there exists a $\mathbf{u} \in U$ satisfying $\mathbf{u}(x_j) = \alpha^j$, $j = 1, ..., k$.*

Remark. It is to be understood that if $n < m$, then (b) is empty.

Remark. A "special case" of this theorem where $m = \infty$ is Theorem 7.3.

Proof. ($\Rightarrow$) If (a) does not hold, then there exists a $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ and $x_1, ..., x_k$ (distinct points in D) with $k \geqslant n/m$, such that

$$\mathbf{u}^*(x_j) = 0, \qquad j = 1, ..., k.$$

Set

$$V = \{(u_1(x_1), ..., u_m(x_1), ..., u_1(x_k), ..., u_m(x_k)) : \mathbf{u} \in U\}.$$

Note that V is a subspace of $\mathbb{R}^{mk}$. Furthermore, since $\mathbf{u}^*|_{I} = \mathbf{0}$, we have

$$\dim V \leqslant n-1 < mk,$$

i.e., V does *not* span $\mathbb{R}^{mk}$.

Now assume $n \geqslant m$ and (b) does not hold. Then there exist points $x_1, ..., x_k$, $k = [n/m]$, such that

$$V = \{(u_1(x_1), ..., u_m(x_1), ..., u_1(x_k), ..., u_m(x_k)) : \mathbf{u} \in U\},$$

does *not* span $\mathbb{R}^{mk}$.

Thus in both cases we have that the V as above does not span $\mathbb{R}^{mk}$. Given any norm on $\mathbb{R}^{mk}$ there will therefore exist some non-zero vector with the zero vector as a best approximant to it from V. As such there exists an $\mathbf{h} \in \mathbb{R}^{mk} \setminus \{\mathbf{0}\}$, which for convenience we write as

$$\mathbf{h} = (h_1(x_1), ..., h_m(x_1), ..., h_1(x_k), ..., h_m(x_k)),$$

such that

$$\min_{\mathbf{u} \in U} \max_{j=1,...,k} \|\mathbf{h}(x_j) - \mathbf{u}(x_j)\|_X = \max_{j=1,...,k} \|\mathbf{h}(x_j)\|_X, \tag{16.1}$$

where we understand that

$$\mathbf{h}(x_j) = (h_1(x_j), ..., h_m(x_j)).$$

Normalize $\mathbf{h}$ so that

$$\max_{j=1,...,k} \|\mathbf{h}(x_j)\|_X = 1. \tag{16.2}$$

Construct $\mathbf{g} \in Y$, i.e., $\mathbf{g}(x) = (g_1(x), ..., g_m(x))$, $g_i \in C(D)$, $i = 1, ..., m$, such that

$$\mathbf{g}(x_j) = \mathbf{h}(x_j), \qquad j = 1, ..., k,$$

and $\|\mathbf{g}(x)\|_X \leqslant 1$ for all $x \in D$, i.e., $\|\mathbf{g}\|_Y = 1$. Such a construction is possible.

We now choose $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ satisfying $\mathbf{u}^*(x_j) = \mathbf{0}$, $j = 1, ..., k$. If (a) holds, then its existence is guaranteed by definition. If $n \geqslant m$ and (b) holds, then since $\dim V < mk \leqslant n = \dim U$, there must also exist such a $\mathbf{u}^*$. We normalize $\mathbf{u}^*$ so that $\|\mathbf{u}^*\|_Y = 1$.

Set

$$\mathbf{f}(x) = \mathbf{g}(x)[1 - \|\mathbf{u}^*(x)\|_X].$$

Note that $\mathbf{f} \in Y$. Since $0 \leqslant 1 - \|\mathbf{u}^*(x)\|_X \leqslant 1$ and

$$\mathbf{f}(x_j) = \mathbf{g}(x_j) = \mathbf{h}(x_j), \qquad j = 1, ..., k, \tag{16.3}$$

we have $\|\mathbf{f}\|_Y = 1$.

For any $\mathbf{u} \in U$ we have from (16.1), (16.2), and (16.3),

$$\begin{aligned} \|\mathbf{f} - \mathbf{u}\|_Y &\geqslant \max_{j=1,...,k} \|(\mathbf{f} - \mathbf{u})(x_j)\|_X \\ &= \max_{j=1,...,k} \|(\mathbf{h} - \mathbf{u})(x_j)\|_X \\ &\geqslant \max_{j=1,...,k} \|\mathbf{h}(x_j)\|_X \\ &= 1. \end{aligned}$$

Furthermore, for $|\alpha| \leqslant 1$ and each $x \in D$,

$$\begin{aligned} \|(\mathbf{f} - \alpha\mathbf{u}^*)(x)\|_X &\leqslant \|\mathbf{f}(x)\|_X + |\alpha| \, \|\mathbf{u}^*(x)\|_X \\ &= \|\mathbf{g}(x)\|_X \, [1 - \|\mathbf{u}^*(x)\|_X] + |\alpha| \, \|\mathbf{u}^*(x)\|_X \\ &\leqslant [1 - \|\mathbf{u}^*(x)\|_X] + |\alpha| \, \|\mathbf{u}^*(x)\|_X \\ &\leqslant 1. \end{aligned}$$

Thus

$$\|\mathbf{f} - \alpha\mathbf{u}^*\|_Y \leqslant 1,$$

and $\alpha\mathbf{u}^*$ is a best approximant to $\mathbf{f}$ from U for all $|\alpha| \leqslant 1$. U is not a unicity space for Y.

($\Leftarrow$) We present, for variety, two different proofs of this direction. We assume that (a) and (b) holds.

(I) Assume U is not a unicity space for Y. Let $\mathbf{f} \in Y$, $\|\mathbf{f}\|_Y = 1$, and $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ be such that $\pm\mathbf{u}^*$ are best approximants to $\mathbf{f}$ from U. Set

$$J = \{x : \|\mathbf{f}(x)\|_X = \|\mathbf{f}\|_Y = 1\}.$$

For any $x \in J$,

$$2\,\|\mathbf{f}(x)\|_X = \|(\mathbf{f} - \mathbf{u}^*)(x)\|_X + \|(\mathbf{f} + \mathbf{u}^*)(x)\|_X.$$

Since X is strictly convex, this implies that for each such x,

$$\mathbf{u}^*(x) = \mathbf{0}.$$

Since J is not empty, this immediately implies, if $n \leqslant m$, that (a) does not hold. Thus we may now assume that $n > m$.

If J contains $k \geqslant n/m$ points, then (a) does not hold. As such we assume that $\{x_1, ..., x_k\} = J$, and $1 \leqslant k < n/m$. Now there exist vectors $\mathbf{d}^1, ..., \mathbf{d}^k \in \mathbb{R}^m$ and $\varepsilon_0 > 0$, such that

$$\|\mathbf{f}(x_j) - \varepsilon\, \mathbf{d}^j\|_X < \|\mathbf{f}(x_j)\|_X = 1, \qquad j = 1, ..., k,$$

for all $0 < \varepsilon \leqslant \varepsilon_0$. Since (b) holds, there exists a $\tilde{\mathbf{u}} \in U$ satisfying

$$\tilde{\mathbf{u}}(x_j) = \mathbf{d}^j, \qquad j = 1, ..., k.$$

By continuity, there is an open neighborhood of each x_j such that

$$\|(\mathbf{f} - \varepsilon \tilde{\mathbf{u}})(x)\|_X < 1$$

on this neighborhood for each $\varepsilon \in (0, \varepsilon_0)$. On the closed (compact) complement of this finite union of neighborhoods,

$$\|\mathbf{f}(x)\|_X < 1 - \eta$$

for some $\eta > 0$. Thus for $\varepsilon > 0$, sufficiently small,

$$\|(\mathbf{f} - \varepsilon \tilde{\mathbf{u}})(x)\|_X < 1$$

for all $x \in D$. That is,

$$\|\mathbf{f} - \varepsilon \tilde{\mathbf{u}}\|_Y < 1$$

which contradicts our assumption that $\mathbf{0}$ is a best approximant to $\mathbf{f}$.

(II) As above, we assume that U is not a unicity space for Y, $\mathbf{f} \in Y$, and $\pm \mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ are best approximants to $\mathbf{f}$ from U.

We let J be as defined, i.e.,

$$J = \{x : \|\mathbf{f}(x)\|_X = \|\mathbf{f}\|_Y\}.$$

Since $\mathbf{0}$ is a best approximant to $\mathbf{f}$ from U, we have from Theorem 16.1 the existence of $\{x_1, ..., x_k\} \subseteq J$, $1 \leqslant k \leqslant n+1$, positive numbers $\{\lambda_j\}_{j=1}^k$, and extremal points $\{\mathbf{h}_j\}_{j=1}^k$ of the unit ball of X^*, such that (1) and (2) of Theorem 16.1 hold. As in (I), we may assume that $1 \leqslant k < n/m$ (for otherwise we contradict (a)). From (b), there exists a $\tilde{\mathbf{u}} \in U$ satisfying

$$\tilde{\mathbf{u}}(x_j) = \mathbf{h}^j, \qquad j = 1, ..., k.$$

Substituting $\tilde{\mathbf{u}}$ in (1) of Theorem 16.1, we get

$$\sum_{j=1}^{k} \sum_{i=1}^{m} \lambda_j \,(h_i^j)^2 = 0.$$

But since $\lambda_j > 0$ and the $\mathbf{h}^j$ are not identically zero, we obtain a contradiction. ∎

Remark. If X is not a strictly convex norm on $\mathbb{R}^m$ then the second part of the proof of Theorem 16.2 need not hold. However, the first part does not depend on this fact. In other words, if U is a unicity space, we must have (a) and (b), where X is any norm on $\mathbb{R}^m$.

Remark. If n/m is an integer, then (a) and (b) are equivalent. This easily follows from the fact that for $k = n/m$, both (a) and (b) are equivalent to the fact that the V in the proof of Theorem 16.2 spans $\mathbb{R}^{mk}$.

We now consider our standard three examples.

EXAMPLE 1. dim $U = 1$. In this case where $U = \text{span}\{\mathbf{u}^*\}$ we have that the unicity space property is equivalent to $Z(\mathbf{u}^*) = \varnothing$. That is, for each $x \in D$ there exists a $j \in \{1, ..., m\}$ such that $u_j^*(x) \neq 0$.

EXAMPLE 2. Tensor Product. Assume $U = U^1 \oplus \cdots \oplus U^m$. Let $k = [n/m]$. Thus $n = km + r$, where $0 \leqslant r < m$.

PROPOSITION 16.3. *U, as above, is a unicity space for Y if and only if r of the $U^1, ..., U^m$ are Haar spaces of dimension $k+1$, and $m-r$ of the $U^1, ..., U^m$ are Haar spaces of dimension k.*

Proof. It is easily seen that (a) and (b) hold for U if and only if for each $i \in \{1, ..., m\}$,

(a′) $|Z(u_i)| < n/m$ for all $u_i \in U^i \setminus \{0\}$

(b′) if $k = [n/m]$, then for every choice of $x_1, ..., x_k$ (distinct) in D, and $\alpha_i^1, ..., \alpha_i^k \in \mathbb{R}$, there exists a $u_i \in U^i$ satisfying $u_i(x_j) = \alpha_i^j$, $j = 1, ..., k$.

If $r = 0$, i.e., $n = km$, then (a′) and (b′) are equivalent. For (a′) to hold it is necessary that each U^i be a subspace of dimension at most k, and if U^i is a subspace of dimension k, then it must be a Haar space. Since $n = \sum_{i=1}^m \dim U^i$, the above implies that each U^i must be a Haar space of dimension k. And in the opposite direction, if each U^i is a Haar space of dimension k, then (a′) and (b′) hold.

Assume $1 \leqslant r < m$. For (a′) to hold, it is necessary that each U^i be a subspace of dimension at most $k+1$, and if U^i is a subspace of dimension $k+1$, then it must be a Haar space. For (b′) to hold, it is necessary that each U^i be a subspace of dimension at least k, and if U^i is a subspace of dimension k, then it must be a Haar space. These facts together imply that each U^i is a Haar space of dimension k or $k+1$. Furthermore, if this is the case, then (a′) and (b′) necessarily hold. ∎

EXAMPLE 3. Simultaneous Approximation. Let $\tilde{U}$ be an n-dimensional subspace of $C(D)$, and

$$U = \{\mathbf{u} = (u, ..., u) : u \in \tilde{U}\}.$$

Assume $m \geqslant 2$ and $n \geqslant 2$. Then U cannot be a unicity space for Y. This follows from the fact that (a) cannot hold. That is, there exists a $u \in \tilde{U} \setminus \{0\}$ with at least $n-1$ zeros in D. Thus $|Z(\mathbf{u})| \geqslant n-1 \geqslant n/m$.

17. $B(p, \infty)$, $1 < p < \infty$

For convenience, we let D be a compact Hausdorff set, and $C(D)$ the space of continuous real-valued functions defined on D. We set

$$\mathbf{f}(x) = (f_1(x), ..., f_m(x)),$$

where each $f_i \in C(D)$, and define the norm Y_p on this space by

$$\|\mathbf{f}\|_{Y_p} = \left(\int_D \left[\max_{i=1,...,m} |f_i(x)|\right]^p dv(x)\right)^{1/p},$$

where v is some finite, positive measure. In this section we assume $1 < p < \infty$. Concerning best approximation in this norm, very little is known. We do have, as a consequence of Theorem 2.3:

PROPOSITION 17.1. *Let U be a linear subspace of Y_p. Then $\mathbf{u}^*$ is a best approximant to $\mathbf{f}$ from U if and only if*

$$\int_D \left[\max_{i=1,...,m} |(f_i - u_i^*)(x)|\right]^{p-1} \times \max_{j \in A(x)} [\operatorname{sgn}((f_j - u_j^*)(x))\, u_j(x)]\, dv(x) \geqslant 0$$

for all $\mathbf{u} \in U$, where

$$A(x) = \{j : |(f_j - u_j^*)(x)| = \max_{i=1,...,m} |(f_i - u_i^*)(x)|\}.$$

The space $B(p, \infty)$ has a similar form to the space $A(p, \infty)$. As a parallel to Proposition 8.2 we have:

PROPOSITION 17.2. *Let U be a finite dimensional subspace of Y_p, $1 < p < \infty$. Assume that for each $\mathbf{u} \in U \setminus \{\mathbf{0}\}$ there exists some $x \in D$ for which $u_i(x) \neq 0$, $i = 1, ..., m$. Then U is a unicity space for Y_p.*

Proof. Assume U is not a unicity space. Let $\mathbf{f} \in Y$ and $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ be such that $\pm \mathbf{u}^*$ are best approximants to $\mathbf{f}$ from U. Now since

$$\|\mathbf{f}\|_{Y_p} = \|\mathbf{f} \pm \mathbf{u}^*\|_{Y_p},$$

it follows from the strict convexity of the L^p-norm that

$$\max_{i=1,\ldots,m} |f_i(x)| = \max_{i=1,\ldots,m} |(f_i - u_i^*)(x)| = \max_{i=1,\ldots,m} |(f_i + u_i^*)(x)|$$

for all $x \in D$. Let

$$J(x) = \{j : |f_j(x)| = \max_{i=1,\ldots,m} |f_i(x)|\}.$$

For $j \in J(x)$, we have $u_j^*(x) = 0$. Thus for each $x \in D$ there exists a j (depending on x) such that $u_j^*(x) = 0$. A contradiction. ∎

One immediate application of the above proposition is the fact that in the problem of Simultaneous Approximation, every U is a unicity space for Y_p.

If $U = \text{span}\{\mathbf{u}^*\}$, i.e., $\dim U = 1$, then the condition given in the above Proposition 17.2 is both necessary and sufficient. This is not difficult to prove. Assume that given each $x \in D$ there exists an i (dependent on x) such that $u_i^*(x) = 0$. Normalize $\mathbf{u}^*$ so that $|u_i^*(x)| \leqslant 1/2$ for all $i = 1, \ldots, m$ and $x \in D$. Set $f_i(x) = 1 - |u_i^*(x)|$, $i = 1, \ldots, m$. Thus, by assumption,

$$\max_{i=1,\ldots,m} |f_i(x)| = 1$$

for all $x \in D$. It now follows that $\alpha \mathbf{u}^*$ are best approximants to $\mathbf{f}$ from U for all $|\alpha| \leqslant 1$.

Unfortunately, while the condition given in Proposition 17.2 is sufficient to ensure that U is a unicity space, it is not necessary. As an example, let D be any connected set, and

$$U = \text{span}\{\mathbf{u}^1, \mathbf{u}^2\}$$

where $\mathbf{u}^1(x) = (1, 1, 1)$, and $\mathbf{u}^2(x) = (1, 0, -1)$. (That is, each component of $\mathbf{u}^1(x)$ and $\mathbf{u}^2(x)$ is a constant function.) U does *not* satisfy the condition given in Proposition 17.2. Now, if U is not a unicity space then there exists an $\mathbf{f}$ and $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ as in the proof of Proposition 17.2. By the argument therein, for each $x \in D$, $u_j^*(x) = 0$ for some $j \in \{1, 2, 3\}$. However, U is of such a form that in this case $u_j^*(x) = 0$ for all $x \in D$, and $u_i^*(x) \neq 0$ for $i \neq j$ and every $x \in D$. Since $u_l^*(x) = 0$ for all $l \in J(x)$ where

$$J(x) = \{j : |f_j(x)| = \max_{i=1,2,3} |f_i(x)|\},$$

it follows that $|f_j(x)| > |f_i(x)|$, $i \neq j$, and all $x \in D$. Since D is connected, we have in addition that f_j is of one strict sign on all of D. Thus, for some ε, sufficiently small, and of the sign of f_j,

$$\|\mathbf{f} - \varepsilon \mathbf{u}^1\|_{Y_p} < \|\mathbf{f}\|_{Y_p}$$

for each $p \in [1, \infty]$. Thus U is a unicity space.

18. $B(1, \infty)$

We assume that D is a compact Hausdorff space and $C(D)$ the space of continuous real-valued functions defined on D. We set

$$\mathbf{f}(x) = (f_1(x), ..., f_m(x)),$$

where $f_i \in C(D)$, $i = 1, ..., m$. We define the norm Y on this space by

$$\|\mathbf{f}\|_Y = \int_D \left[\max_{i=1,...,m} |f_i(x)| \right] d\mu(x),$$

where μ is some finite, positive measure. There is very little known about this space. As an application of Theorem 14.1, we have:

PROPOSITION 18.1. *Let U be a finite dimensional subspace of Y. Then $\mathbf{u}^*$ is a best approximant to $\mathbf{f}$ from U if and only if*

$$-\int_{D \setminus Z(\mathbf{f} - \mathbf{u}^*)} [\max_{j \in A(x)} \operatorname{sgn}((f_j - u_j^*)(x_j))\, u_j(x)]\, d\mu(x)$$
$$\leqslant \int_{Z(\mathbf{f} - \mathbf{u}^*)} \|\mathbf{u}(x)\|_\infty\, d\mu(x)$$

for all $\mathbf{u} \in U$, where

$$A(x) = \{j : |(f_j - u_j^*)(x)| = \max_{i=1,...,m} |(f_i - u_i^*)(x)|\}$$

and

$$\|\mathbf{u}(x)\|_\infty = \max_{i=1,...,m} |u_i(x)|.$$

Unfortunately little seems to be known about characterizing unicity spaces in this norm. We can show, paralleling one half of Theorem 14.5, that if U is a unicity space for Y, then $\mathbf{0}$ cannot be a best approximant from U to any $\mathbf{g} \in U^* \setminus \{\mathbf{0}\}$. However, there is no reason to suppose that the converse is valid, and as such the result loses much of its relevence.

19. $B(\infty, 1)$

We assume that D is a compact Hausdorff space and $C(D)$ the space of continuous real-valued functions defined on D. We set

$$\mathbf{f}(x) = (f_1(x), ..., f_m(x)),$$

where $f_i \in C(D)$, $i = 1, ..., m$. We define the norm Y on this space by

$$\|\mathbf{f}\|_Y = \max_{x \in D} \sum_{i=1}^{m} |f_i(x)|.$$

As an application of Theorem 16.1, we have the following characterization of best approximants.

THEOREM 19.1. *Let U be an n-dimensional subspace of Y. Then $\mathbf{u}^*$ is a best approximant to $\mathbf{f}$ from U if and only if for some k, $1 \leqslant k \leqslant n+1$, there exist positive numbers $\{\lambda_j\}_{j=1}^k$, $\varepsilon_{ij} \in \{-1, 1\}$, $i = 1, ..., m$; $j = 1, ..., k$, and points $\{x_j\}_{j=1}^k$ in A, where*

$$A = \left\{x : \sum_{i=1}^{m} |(f_i - u_i^*)(x)| = \|\mathbf{f} - \mathbf{u}^*\|_Y\right\},$$

satisfying

(1) $\varepsilon_{ij} = \operatorname{sgn}((f_i - u_i^*)(x_j))$ *if* $(f_i - u_i^*)(x_j) \neq 0$

(2) $\sum_{j=1}^{k} \sum_{i=1}^{m} \lambda_j \varepsilon_{ij} u_i(x_j) = 0$

for all $\mathbf{u} \in U$.

From the Remark immediately after Theorem 16.2, we have that if U is a unicity space for Y, then (a) and (b) of Theorem 16.2 must hold. From this fact it follows, as in Example 3 of Section 16, that in the problem of Simultaneous Approximation for $m \geqslant 2$, $n \geqslant 2$, U is not a unicity space for Y. However, conditions (a) and (b) are *not* in general sufficient to guarantee that U be a unicity space. This we note as a consequence of this next result.

PROPOSITION 19.2. *If U is not a unicity space, then there exists a $\mathbf{u}^* \in U \backslash \{\mathbf{0}\}$, $x^* \in D$, and $\varepsilon_i \in \{-1, 1\}$, $i = 1, ..., m$, such that*

$$\sum_{i=1}^{m} \varepsilon_i u_i^*(x^*) = 0. \tag{19.1}$$

Furthermore if $\dim U = 1$ $(U = \operatorname{span}\{\mathbf{u}^*\})$, *then the existence of such an* x^* *and* $\{\varepsilon_i\}$ *implies that* U *is not a unicity space.*

Proof. Assume U is not a unicity space. Let $\mathbf{f} \in Y$ and $\mathbf{u}^* \in U \setminus \{\mathbf{0}\}$ be such that $\pm\mathbf{u}^*$ are best approximants to $\mathbf{f}$ from U. A simple calculation shows that if

$$\|\mathbf{f}\|_Y = \sum_{i=1}^{m} |f_i(x^*)|$$

then

$$2\,|f_i(x^*)| = |(f_i - u_i^*)(x^*)| + |(f_i + u_i^*)(x^*)| \tag{19.2}$$

for $i = 1, ..., m$, and

$$\|\mathbf{f} \pm \mathbf{u}^*\|_Y = \sum_{i=1}^{m} |(f_i \pm u_i^*)(x^*)|.$$

From (19.2),

$$|f_i(x^*)| \geqslant |u_i^*(x^*)|, \qquad i = 1, ..., m.$$

Set $\varepsilon_i = \operatorname{sgn}(f_i(x^*))$ if $f_i(x^*) \neq 0$. If $f_i(x^*) = 0$, then $u_i^*(x^*) = 0$ and ε_i may be arbitrarily chosen in $\{-1, 1\}$. Now

$$\begin{aligned}
\sum_{i=1}^{m} \varepsilon_i f_i(x^*) = \sum_{i=1}^{m} |f_i(x^*)| &= \|\mathbf{f}\|_Y = \|\mathbf{f} \pm \mathbf{u}^*\|_Y \\
&= \sum_{i=1}^{m} |(f_i \pm u_i^*)(x^*)| = \sum_{i=1}^{m} \varepsilon_i (f_i \pm u_i^*)(x^*) \\
&= \sum_{i=1}^{m} \varepsilon_i f_i(x^*) \pm \sum_{i=1}^{m} \varepsilon_i u_i^*(x^*).
\end{aligned}$$

Thus

$$\sum_{i=1}^{m} \varepsilon_i u_i^*(x^*) = 0.$$

It remains to prove the latter half of the proposition. Assume $U = \operatorname{span}\{\mathbf{u}^*\}$, and (19.1) holds. Set

$$f_i(x) = \varepsilon_i \left[c - |u_i^*(x) - u_i^*(x^*)|\right],$$

where $c > 3 \|u_i\|_\infty$ for all i (see the proof of Proposition 10.3). Note that for any $x \in D$

$$|f_i(x)| \leqslant |f_i(x^*)| = c,$$

for each $i = 1, ..., m$. Thus

$$\|\mathbf{f}\|_Y = \sum_{i=1}^{m} |f_i(x^*)| \ (= mc).$$

Applying Theorem 19.1 with $k = 1$ (condition (2) therein is given by (19.1)), we have that $\mathbf{0}$ is a best approximant to $\mathbf{f}$ from U. Now $\varepsilon_i (f_i - u_i^*)(x) \geqslant 0$ for each $x \in D$ and $i = 1, ..., m$ since $c > 3 \|u_i\|_\infty$. Furthermore,

$$\begin{aligned}|(f_i - u_i^*)(x)| &= c - |u_i^*(x) - u_i^*(x^*)| - \varepsilon_i u_i^*(x) \\ &\leqslant c - \varepsilon_i u_i^*(x^*) \\ &= \varepsilon_i f_i(x^*) - \varepsilon_i u_i^*(x^*) \\ &= |(f_i - u_i^*)(x^*)|.\end{aligned}$$

Thus,

$$\begin{aligned}\|\mathbf{f} - \mathbf{u}^*\|_Y &= \sum_{i=1}^{m} |(f_i - u_i^*)(x^*)| = \sum_{i=1}^{m} \varepsilon_i (f_i - u_i^*)(x^*) \\ &= \sum_{i=1}^{m} \varepsilon_i f_i(x^*) = \sum_{i=1}^{m} |f_i(x^*)| = \|\mathbf{f}\|_Y,\end{aligned}$$

and $\mathbf{u}^*$ is also a best approximant to $\mathbf{f}$. ∎

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